THE UNIVERSITY OF QUEENSLAND

AUSTRALIA

THE $K(\pi, 1)$ CONJECTURE FOR ARTIN BRAID GROUPS

Honours Thesis

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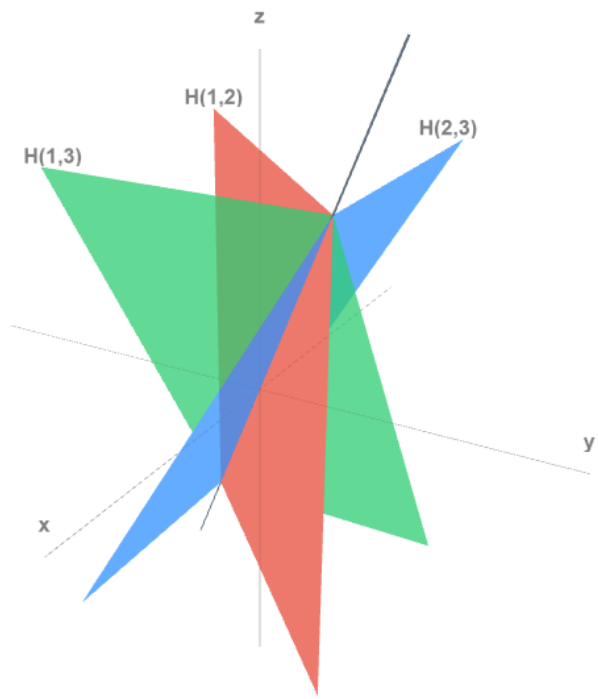

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1. INTRODUCTION

1.1. **Overview.** In this thesis, we provide an accessible introduction to the $K(\pi, 1)$ conjecture for Artin braid groups, and outline Deligne's contribution to this problem.

The central objects of the $K(\pi, 1)$ conjecture are constructed from Coxeter groups. Coxeter groups are discrete groups generated by reflections through hyperplanes. The most familiar examples are the symmetric S_n and dihedral $I_2(n)$ groups and, more generally, the Weyl groups.

From a Coxeter group W we can define an *Artin braid group* B_W and a *configuration space* Y_W. The group B_W has the same generators and relations as W but the relations of the form $s^2 = 1$ are omitted. The space Y_W is a manifold constructed from the reflection hyperplanes of W. In the S_n case, its Artin braid group is the classical braid group $\mathcal{B}_n$ on n strands, and its configuration space is the space Y_n of all n-tuples of distinct points in $\mathbb{C}$.

The Coxeter group W acts freely and properly discontinuously on Y_W, and so its orbit space Y_W/W is a manifold. The $K(\pi, 1)$ conjecture for Artin braid groups asserts the following.

Conjecture ($K(\pi, 1)$ conjecture)**.**

$$\pi_i(Y_W/W) \simeq \begin{cases} B_W & \textit{if } i = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

A space whose higher homotopy groups are all trivial is called a $K(\pi, 1)$ space. This implies B_W is torsion free.

As noted by Brieskorn [6], the conjecture was proposed by Tits who was motivated by the prior work of Arnol'd, Brieskorn, Fadell, Fox, Hurwitz and Neuwirth [2, 19, 6, 13, 14]. For finite W, the conjecture was solved by Brieskorn [6] and Deligne [10] in the 1970s. For affine Weyl groups, the conjecture was settled by Paolini and Salvetti [25] in the 2020s. For general Coxeter groups, the conjecture remains open, although significant progress has recently been made by Huang and Hoda [16, 17].

In §1.2, we begin by defining the classical braid groups $\mathcal{B}_n$, whose presentations closely resemble those of the symmetric groups. In fact, we see that $\mathcal{B}_n$ arises as a group extension of S_n in §1.3. The relationship between $\mathcal{B}_n$ and the fundamental group of the space Y_n of n-tuples of distinct points in $\mathbb{C}$ is discussed in §1.4. Moreover, we discuss how these spaces are intimately connected to the hyperplane arrangements associated with S_n.

In §1.5, the Coxeter groups are introduced and we use them to define the Artin braid groups. The associated configuration spaces Y_W are then defined in §1.6. In §1.7 we present the $K(\pi, 1)$ conjecture and discuss, in more detail, its historical context and recent progress towards its solution. We finish with a detailed overview of the goals and content of this thesis in §1.8.

1.2. **The classical braid groups $\mathcal{B}_n$.** We start by considering an intuitive geometric object. With n disjoint strands, we join points in $\{(i, 0, 0) \in \mathbb{R}^3 : 1 \leq i \leq n\}$ to points in $\{(i, 0, 1) \in$

$\mathbb{R}^3 : 1 \leq i \leq n\}$. Taking the union of these strands, we obtain a "braid-like" structure (see Figure 1).

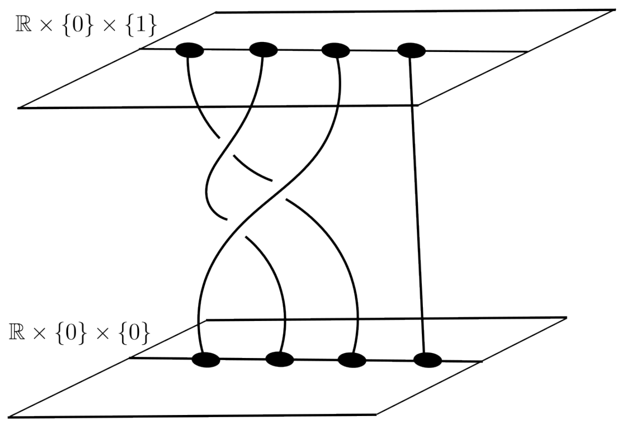

FIGURE 1. Braid-like structure in $\mathbb{R}^3$.

Up to continuous deformations of the strands that fix endpoints, we say that such an arrangement of strands is a **braid** on n strands. Braids arise naturally in several areas of mathematics. Historically, they appeared implicitly in Hurwitz's 1891 study of monodromy of Riemann surfaces [19, 22]. They also play a central role in knot theory, where they are related to knots by a celebrated theorem of Alexander.

Theorem 1 (Alexander [1], Main Theorem)**.** *Every knot or link is obtained by connecting corresponding upper and lower endpoints of some braid (see Figure 2).*

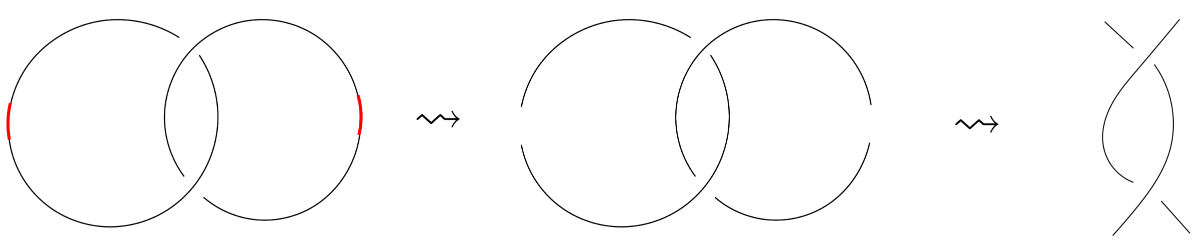

FIGURE 2. Braid representation of the Hopf Link.

Given two braids, there is a natural way to compose them to produce another (see Figure 3).

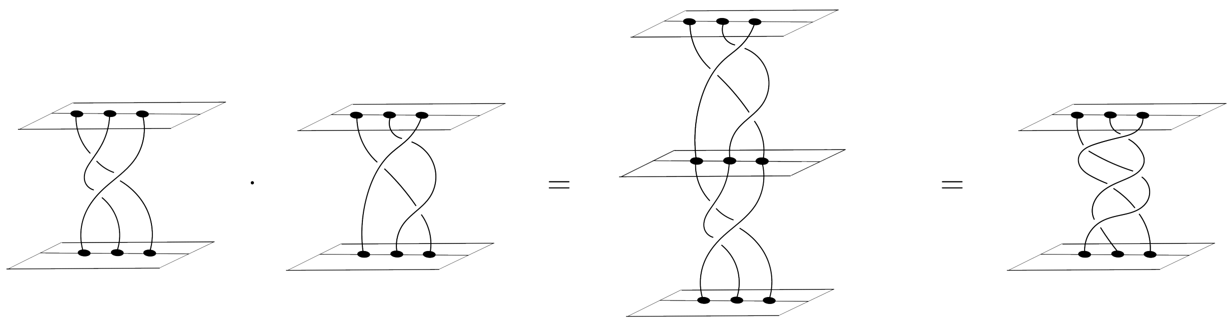

FIGURE 3. Multiplication of braids.

Braids on n strands form a group under composition. The identity element is the braid consisting of vertical lines connecting the points $(i, 0, 0)$ and $(i, 0, 1)$ for all $1 \leq i \leq n$. To see that every braid has an inverse, one can show that every braid can be broken up into a finite product of "elementary braids" (see Proposition 5 for more details) of the following form (Figure 4).

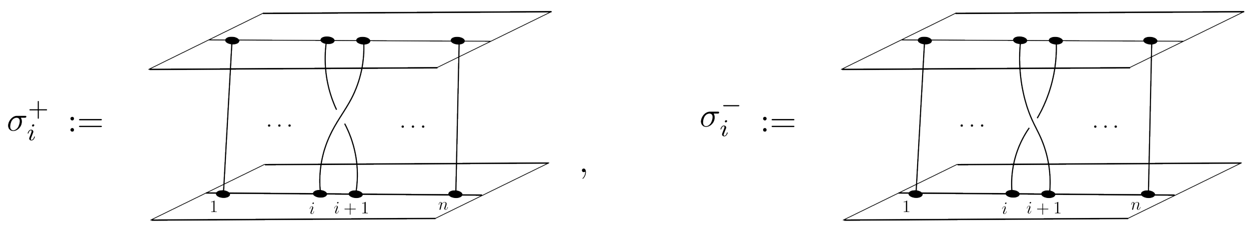

FIGURE 4. Braids with a single crossing.

From such a finite decomposition, there is a natural way to construct an inverse braid using the fact that σ_i^+ and σ_i^- are inverses of each other (see Figure 5).

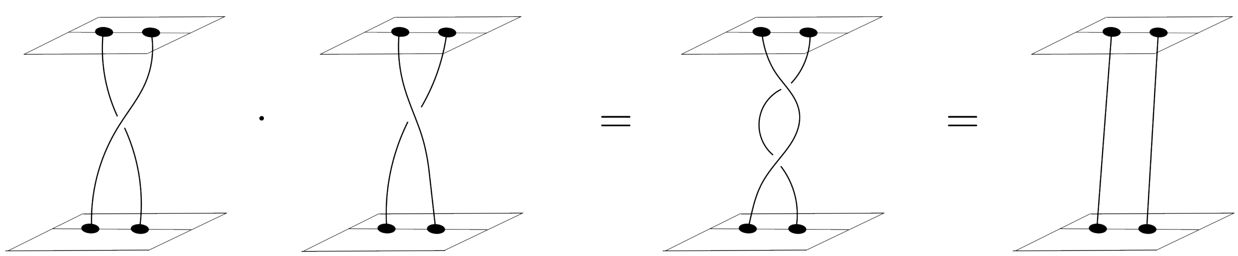

FIGURE 5. $\sigma_i^+ \cdot \sigma_i^- = 1$.

We call the collection of braids on n strands under composition the **classical braid group** $\mathcal{B}_n$ on n strands. From a geometric point of view, braids can become convoluted to study. However, on 2 strands, one can see that there are infinitely many braids generated by σ_1^+ (see Figure 6). That is, $\mathcal{B}_2 \simeq \langle \sigma_1^+ \rangle \simeq \mathbb{Z}$.

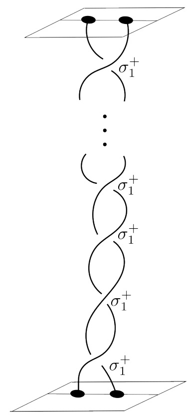

FIGURE 6. $\mathcal{B}_2 \simeq \mathbb{Z}$.

1.3. **Relation to the symmetric group** S_n. There is a geometric relationship between braids on n strands and permutations of an n-set. Given a braid, one can project its strands onto the plane determined by the equation $y = 0$. The projection forms a strand diagram, representing a permutation of n points (see Figure 7).

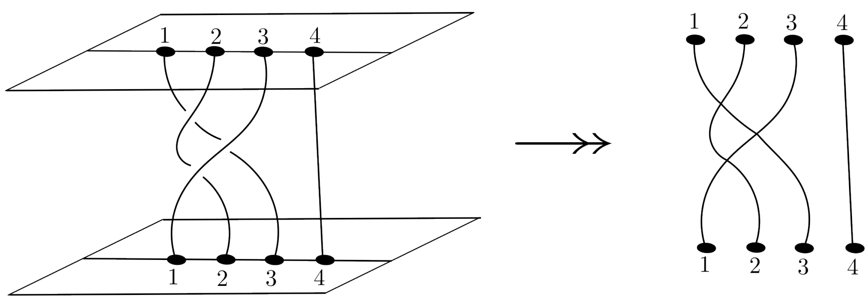

FIGURE 7. Projection map $\mathcal{B}_n \twoheadrightarrow S_n$.

The projection induces a homomorphism $\mathcal{B}_n \to S_n$ whose kernel defines an important subgroup called the **pure braid group** $P\mathcal{B}_n$, consisting of all braids whose strands join the points $(i, 0, 0)$ to $(i, 0, 1)$ for all $1 \leq i \leq n$ (for example, Figure 8).

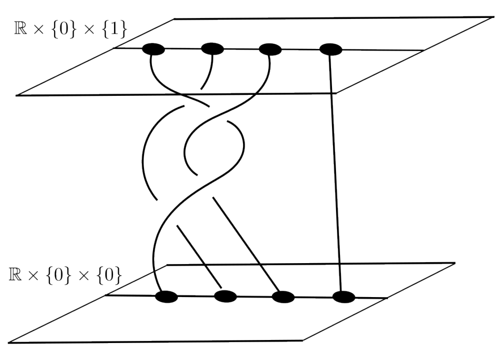

FIGURE 8. Example of a pure braid.

Another perspective on the relationship between the braid groups and the symmetric groups is to compare their group presentations. In 1896, Moore was able to show that the symmetric group S_n could be generated by $n-1$ generators in the following way.

Theorem 2 (Moore [23], Theorem A)**.**

$$S_n = \left\langle s_1, s_2, \cdots s_{n-1} : \begin{array}{cc} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{for } 1 \le i < n-1 \\ s_i s_j = s_j s_i & \text{for } |i-j| > 1 \\ s_i^2 = 1 & \text{for all } i \end{array} \right\rangle$$

Moreover, in 1925, Emil Artin formalised the braid groups by demonstrating that they admit the following group presentation on $n-1$ generators.

Theorem 3 (Artin [3], Satz 1)**.**

$$\mathcal{B}_n = \left\langle s_1, s_2, \cdots s_{n-1} : \begin{array}{cc} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \text{for } 1 \le i < n-1 \\ s_i s_j = s_j s_i & \text{for } |i-j| > 1 \end{array} \right\rangle.$$

The group presentation of S_n differs from the presentation of $\mathcal{B}_n$ only by the additional family of relations that require its generators to be self-inverse. As such, the braid groups arise as group extensions of S_n

$$1 \longrightarrow P\mathcal{B}_n \longrightarrow \mathcal{B}_n \longrightarrow S_n \longrightarrow 1.$$

1.4. **The space Y_n.** Hurwitz was originally interested in computing the fundamental group of the topological manifold

$$Y_n := \mathbb{C}^n - \{\text{Zero locus of the function } \Delta\}$$

where $\Delta : \mathbb{C}^n \to \mathbb{C}$ given by

$$\Delta(z_1, \ldots, z_n) = \prod_{i<j} (z_i - z_j).$$

He implicitly showed that

$$P\mathcal{B}_n \simeq \pi_1(Y_n). \tag{1.1}$$

The space Y_n is the space of n-tuples of distinct points in $\mathbb{C}$

$$Y_n = \mathbb{C}^n - \bigcup_{i \neq j} \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_i = z_j\}.$$

The symmetric group acts freely on Y_n by permuting coordinates. As an immediate corollary (see Corollary 13)

$$\mathcal{B}_n \simeq \pi_1\left({}^{Y_n}/_{S_n} \right). \tag{1.2}$$

That is, the braid group $\mathcal{B}_n$ arises as the fundamental group of the space of n-sets from points in $\mathbb{C}$.

The interpretation of the classical braid groups as the fundamental group of Y_n/S_n appears not to have been widely recognised in the literature between the 1920s and the 1960s. This may have been a consequence of Hurwitz's work arising in the distant context of Riemman surfaces. Regardless, in 1962 this interpretation was rediscovered by Fox and Neuwirth [14]. That same year, Fadell and Neuwirth further discovered that

$$\pi_i \left({}^{Y_n}/{}_{S_n} \right) \simeq 0$$

for all $i \geq 2$ [13]. This implies that $\mathcal{B}_n$ is torsion free (see §2.6 for more details).

1.5. **Coxeter and Artin braid groups.** Motivated by the discovery that the braid groups could be studied in terms of a configuration space Y_n, the French mathematician Jacques Tits conjectured that this phenomenon could be extended to a wider class of objects [6]. To understand this insight, we must view the symmetric group as a group generated by reflections.

Discrete groups generated by reflections were formalised by H. S. M. Coxeter in 1934 [8]. He defined the **Coxeter groups** to be the abstract groups W admitting a group presentation of the form

$$W = \left\langle s_1, s_2, \cdots s_n : \begin{array}{ll} (s_i s_j)^{m(i,j)} = 1 & \text{for } i \neq j \\ s_i^2 = 1 & \text{for all } i \end{array} \right\rangle$$

such that $m(i,j) \in \{2, 3, \cdots\}$ where some generators may be unrelated.

We define the associated **Artin braid group** B_W to be

$$B_W := \left\langle s_1, s_2, \cdots s_n : \ (s_i s_j)^{m(i,j)} = 1 \quad \text{for } i \neq j \ \right\rangle.$$

Thus, we obtain a surjection $\mathcal{B}_W \twoheadrightarrow W$ whose kernel we define to be the **pure Artin braid group** PB_W. That is, the Artin braid group B_W arises as a group extension of W

$$1 \longrightarrow PB_W \longrightarrow B_W \longrightarrow W \longrightarrow 1.$$

Note that B_W is always an infinite group. We say that B_W has **finite type** when W is finite, and **infinite type** otherwise.

1.6. **Configuration spaces Y_W.** Let H_{ij} denote the hyperplane in $\mathbb{R}^{n-1}$ defined by the equation $x_i = x_j$. One can show Y_n is homotopy equivalent to the space

$$Y_n \simeq \mathbb{C}^n - \bigcup_{i \neq j} H_{ij}^{\mathbb{C}}$$

where $H_{ij}^{\mathbb{C}} := H_{ij} \otimes \mathbb{C}$ (see §4.1). Since every Coxeter group has an associated arrangement of hyperplanes $\mathcal{H}_W$ in some real vector space V, we can define a space analogous to Y_n for arbitrary W. If W is finite, we define

$$Y_W := V^{\mathbb{C}} - \bigcup_{H \in \mathcal{H}_W} H^{\mathbb{C}}$$

to be the **configuration space** of W.

When W is infinite, the set $\mathcal{H}_W$ is an infinite arrangement of hyperplanes, where a sequence of hyperplanes can accumulate towards a limiting hyperplane. To avoid the pathological consequences of this limiting behaviour we define the **Tits cone** of V to be

$$\mathcal{T} := \bigcup_{w \in W} w\overline{C_0}$$

where C_0 is an open simplicial cone of V^* called a **fundamental chamber** (for more details, see §4.3). We then define

$$Y_W := \mathcal{T} \times \mathcal{T} - \bigcup_{H \in \mathcal{H}_W} H^{\mathbb{C}}$$

to be the **configuration space** of W, where $\mathcal{H}_W$ is identified with a hyperplane arrangement in V^*. In §4.3 we show that

$$\mathcal{T} = V \iff W \text{ is finite}$$

and so both constructions are homeomorphic in the finite case.

1.7. **The** $K(\pi, 1)$ **conjecture.** It is natural to ask whether the observations made by Hurwitz, Neuwirth, Fadell and Fox about the classical braid groups also apply to the Artin braid groups.

Conjecture (The $K(\pi, 1)$ conjecture)**.** *For an Artin braid group* B_W*, the following holds*

$$(C1) : B_W \simeq \pi_1\left(Y_W / W\right) \quad \textit{and} \quad (C2) : \pi_i\left(Y_W / W\right) \simeq 0$$

for all $i \geq 2$*.*

For the classical braid groups, (C1) was originally proved by Hurwitz in 1891 [19], but rediscovered by Fox and Neuwirth in 1962 [14]. That same year, Fadell and Neuwirth published their proof [13] that (C2) held for $\mathcal{B}_n$. In 1969, Arnol'd [2] computed the cohomology ring of Y_n, revealing rich topological structure and providing further motivation for the study of Artin braid groups.

For finite type Artin braid groups, Brieskorn showed that (C1) held and made some progress on (C2) in 1971 [6]. The following year, Deligne proved that (C2) held for all finite type Artin braid groups by showing that the universal covering space of Y_W is contractible. Deligne did this by relating the universal covering space to a simplicial complex built up of contractible simplices, now called the **Deligne complex** I [10] (see §6.1).

In 1983, van der Lek in his Ph.D thesis proved that (C1) held for any Artin braid group [21]. His argument involved developing what is known as the category of **pregalleries** (which we discuss in §5.2). In 2021, Paolini and Salvetti extended $(C2)$ to all affine Coxeter groups [25, 26]. Delucchi, in collaboration with the previous authors, in 2024 extended $(C2)$ to all Coxeter groups of rank 3, which include a large supply of hyperbolic Coxeter groups (a well-known family of infinite Coxeter groups for which the conjecture remains open) [12]. In 2026, Huang released a preprint in which they reduce the $K(\pi, 1)$ conjecture to statements about Artin braid groups whose Coxeter graphs (see §3.1) are trees [16]. This has already been used by Hoda and the previous author to show that the conjecture holds for a wider class of Artin braid groups [17].

1.8. **Goals and structure of thesis.** The goal of this thesis is to understand some of the major themes behind the $K(\pi, 1)$ conjecture and to communicate those ideas in an accessible form. We assume only an introductory background in algebraic topology.

To accomplish this goal, we introduce the geometric and combinatorial structures that play a central role in the $K(\pi, 1)$ conjecture, namely braid groups, Coxeter groups, hyperplane arrangements and galleries. The following diagram illustrates the logical dependency of the chapters in this thesis. If Chapter $i \longrightarrow$ Chapter j, then Chapter i should be read before Chapter j.

$$\begin{tikzcd}
\begin{array}{c}\text{Chapter 1}\\ \text{Introduction}\end{array} & \begin{array}{c}\text{Chapter 2}\\ \text{Braid groups}\\ \mathcal{B}_n\end{array} \arrow[d] & \begin{array}{c}\text{Chapter 3}\\ \text{Coxeter groups}\\ W\end{array} \arrow[dl] \\
& \begin{array}{c}\text{Chapter 4}\\ \text{Configuration spaces}\\ Y_W\end{array} \arrow[d] & \\
& \begin{array}{c}\text{Chapter 5}\\ \text{Galleries}\\ (V, \mathcal{H})\end{array} \arrow[d] & \\
& \begin{array}{c}\text{Chapter 6}\\ \text{Outline of Deligne's theorem}\end{array} &
\end{tikzcd}$$

We now consider the contents of each chapter in more detail.

Chapter 2: Braid groups. We introduce the braid groups $\mathcal{B}_n$ and relate them to the symmetric groups S_n and configuration spaces Y_n. We then outline the arguments of Neuwirth, Fox and Fadell showing that Y_n/S_n is $K(\mathcal{B}_n, 1)$. We then discuss the implications of this result on the group cohomology of $\mathcal{B}_n$ and prove that $\mathcal{B}_n$ must be torsion free.

Chapter 3: Coxeter groups. We introduce the basic theory of Coxeter groups W. The Coxeter systems are defined and properties of their reduced expressions are explored. We introduce the reflection representation of W, prove that it is faithful, and thereby realise W as a group generated by reflections. The reflections of W are then related to an associated root system and hyperplane arrangement. We outline the proof of the classification of finite Coxeter groups and finish by generalising the braids groups via the Artin braid groups.

Chapter 4: Configuration spaces. The space Y_n is related to a space determined by the hyperplane arrangement of S_n. We then use this observation to define configuration spaces Y_W for Coxeter groups, discussing the additional subtleties that arise when W is infinite. The chapter culminates in the statement of the $K(\pi, 1)$ conjecture and an overview of the work of Brieskorn and Deligne that motivates the later chapters.

Chapter 5: Galleries. We seek to relate the combinatorial features of the hyperplane complement of W to the topology of Y_W. To do this, we introduce pregalleries and the fundamental groupoid, motivating the theory of galleries of hyperplane arrangements. In particular, we observe the special properties of galleries of simplicial hyperplane arrangements.

Chapter 6: An outline of Deligne's theorem. In this chapter, we sketch Deligne's proof of (C2) for Artin braid groups of finite type. In doing so, we construct the Deligne complex I and introduce the theory of simplicial collapse and Leray's theorem.

1.9. **Acknowledgements.** I would like to begin by acknowledging the Traditional Owners, the Turrbal and Jagera peoples, of the unceded land on which my university education took place.

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2. BRAID GROUPS

Central to the formation of the $K(\pi, 1)$ conjecture was the discovery that so called braid groups arise as the fundamental groups of an important class of configuration spaces. In this section, we introduce the braid groups and relate them to the symmetric groups (§2.1-2.2). In §2.3 we introduce the ordered configuration space of n points $\mathcal{C}_n(M)$ over a topological manifold M. In the following sections (§2.4-2.5) we outline the arguments of Neuwirth, Fox and Fadel which relate $\mathcal{C}_n(\mathbb{C})$ to $\mathcal{B}_n$ by showing that it is $K(\mathcal{B}_n, 1)$. In the final section §2.6 we discuss the cohomological upshot of these results which imply that the braid groups have the cohomology of a manifold. In particular, this implies that the braid groups are torsion free.

2.1. **The braid group $\mathcal{B}_n$.** The braid groups were explicitly introduced in 1925 by Emil Artin in his paper "*Theorie der Zöpfe*" [3], although they were implict in the earlier work of Adolf Hurwitz on monodromy in 1891 [19].

We introduce the braid groups by studying the algebraic properties of the geometric notion of braids. If $I = [0, 1]$, then a braid is an object embedded in $\mathbb{R}^2 \times I$, constructed as follows: Take n distinct points along the line $\mathbb{R} \times \{0\} \times \{0\}$ and directly above on the line $\mathbb{R} \times \{0\} \times \{1\}$ and join the points from the top line and the bottom line by disjoint curves such that the projection $\mathbb{R}^2 \times I \to \mathbb{R} \times \{0\} \times I$ restricted to each curve is a homeomorphism (Figure 9).

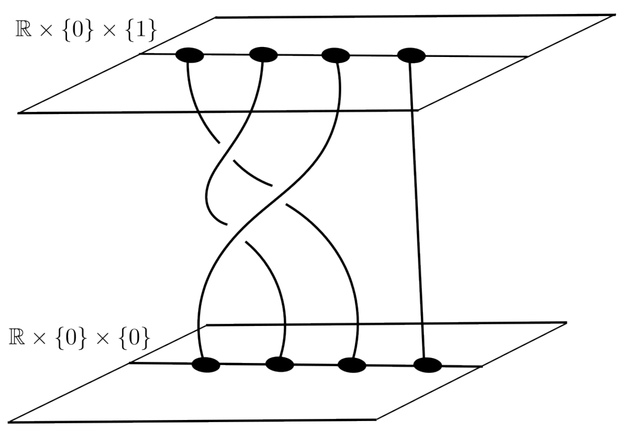


FIGURE 9. Geometric braid on 4 strands.

In particular, at each t in I, the intersection of the plane $z = t$ with the curves in $\mathbb{R}^2 \times I$ is a discrete set of points indexing the curves. The disjoint union of these curves, embedded in $\mathbb{R}^2 \times I$, is called a **geometric braid** on n strands. Given two geometric braids b and b' on n strands, if there exists a continuous map $F : b \times [0, 1] \to \mathbb{R}^2 \times I$ such that $F(b, 0) = b$, $F(b, 1) = b'$ and for every $t \in [0, 1]$, $F(b, t)$ is a geometric braid, then we say b and b' are isotopic as geometric braids. We call an isotopy class of geometric braids on n strands a **braid on n strands**.

Let $\mathcal{B}_n$ be the collection of braids on n strands. We can define a multiplication on $\mathcal{B}_n$ in the following way: Given β_1 and β_2 in $\mathcal{B}_n$ and class representatives b_1 and b_2, we define the product $\beta_1\beta_2$ to be the isotopy class of the geometric braid obtained by stacking b_2 on top of b_1 and rescaling vertically so that the top endpoints lie on the line $\mathbb{R} \times \{0\} \times \{1\}$ (see Figure 10).

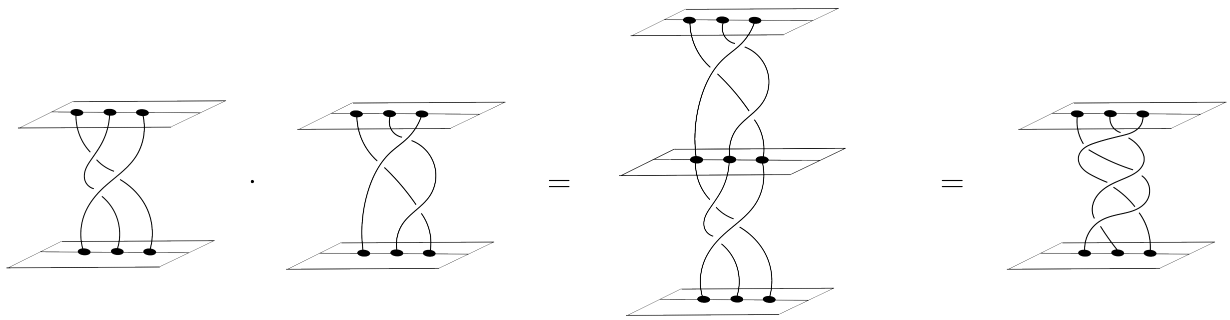


FIGURE 10. Multiplication of braids.

One can easily verify that the multiplication of braids in $\mathcal{B}_n$ is well-defined. There is also a natural identity element 1_n being the isotopy class of the geometric braid whose ith strand is parameterised by $\varphi : I \to \mathbb{R}^3$, $t \mapsto (i, 0, t)$, for all $1 \leq i \leq n$. In fact, it is a result of Artin (Proposition 5) that $\mathcal{B}_n$ forms a group under multiplication called the **classical braid group** $\mathcal{B}_n$ (or simply the **braid group** on n strands).

Technical Remark 4. Given a geometric braid b, let $\mathcal{D}_b$ be its image under the projection $\mathbb{R}^2 \times I \to \mathbb{R} \times \{0\} \times I$ (see Figure 11). We define a **crossing** of b to be a point of intersection of two strands in $\mathcal{D}_b$. In general, the crossings of a geometric braid may not be isolated, however, it is a consequence of the transversality theorem in differential topology that there exists a geometric braid b' isotopic to b such that the preimage of every intersection point in $\mathcal{D}_{b'}$ has at most two elements, and any two strands that intersect at a point, intersect transversely. This implies that b' has isolated crossings, and hence by compactness of the interval I, that b' has finitely many crossings.

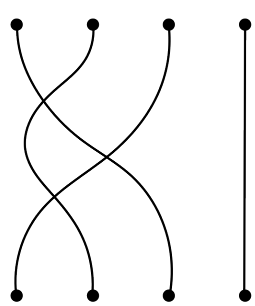
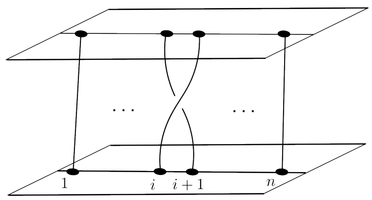


FIGURE 11. Image of geometric braid in Figure 9 under $\mathbb{R}^2 \times I \to \mathbb{R} \times \{0\} \times I$.

Proposition 5. *Every braid has a multiplicative inverse. Therefore, $\mathcal{B}_n$ forms a group under multiplication.*

Proof. For each $1 \leq i \leq n$, we define the geometric braids σ_i^+ and σ_i^- on n strands in Figure 12.

$\sigma_i^+ :=$ 

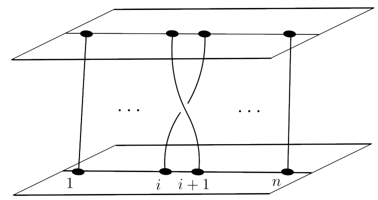
, $\sigma_i^- :=$ 

FIGURE 12. Braids with one crossing.

Let $[\sigma_i^+]$ and $[\sigma_i^-]$ denote the isotopy classes of σ_i^+ and σ_i^- respectively, then we can easily verify that $[\sigma_i^+][\sigma_i^-] = [\sigma_i^-][\sigma_i^+] = 1_n$. By Remark 4, there exists a class representative b of β with finitely many isolated crossings. Hence, we can construct a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of I such that in each region $\mathbb{R} \times [t_{j-1}, t_j]$ of D_b there is exactly one crossing. Thus, we have a sequence of points $1 \leq i_j \leq n$ with $j \in \{0, \ldots, k\}$ such that $\beta = [\sigma_{i_0}^{\varepsilon_0}][\sigma_{i_1}^{\varepsilon_1}] \cdots [\sigma_{i_k}^{\varepsilon_k}]$ where $\varepsilon_j \in \{+, -\}$. Let $\beta^{-1} := [\sigma_{i_k}^{-\varepsilon_k}] \cdots [\sigma_{i_1}^{-\varepsilon_1}][\sigma_{i_0}^{-\varepsilon_k}]$, where $-\varepsilon_{i_j} := -$ if and only if $\varepsilon_{i_j} = +$. Then

$$\beta\beta^{-1} = [\sigma_{i_0}^{\varepsilon_0}][\sigma_{i_1}^{\varepsilon_1}] \cdots [\sigma_{i_k}^{\varepsilon_k}][\sigma_{i_k}^{-\varepsilon_k}] \cdots [\sigma_{i_1}^{-\varepsilon_1}][\sigma_{i_0}^{-\varepsilon_0}] = 1_n$$

and similarly $\beta^{-1}\beta = 1_n$. $\square$

Example 6. We compute $\mathcal{B}_1$ and $\mathcal{B}_2$. The braid group on one strand $\mathcal{B}_1$ has one isotopy class, namely, the class containing the trivial geometric braid. As such, $\mathcal{B}_1$ is the trivial group.

To compute $\mathcal{B}_2$, we refer to the proof of Proposition 5 which tells us that $\mathcal{B}_2$ is generated by the simple crossing $[\sigma_1^+]$ on two strands. Intuitively, $[\sigma_1^+]$ has infinite order since $(\sigma_1^+)^k$ forms a braid with k crossings that do not "untangle" (see Figure 13).

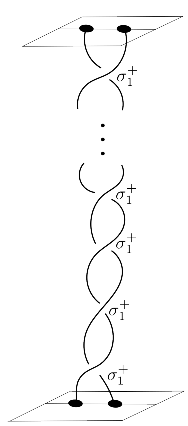


FIGURE 13. The generator σ_1^+ has infinite order.

As such, $\mathcal{B}_2$ is an infinite cyclic group and forms a model of the group of integers $\mathbb{Z}$ as braids.

The advantage of working with geometric braids is that they are intuitive geometric objects: Two geometric braids are the same if I can imagine "pulling" and "interweaving" one braid into the other. However, the geometric picture can become convoluted very quickly. It would be helpful to view the braid group more algebraically, in terms of generators and relations. The following theorem by Artin achieves exactly this.

Theorem 7 (Artin [3], Satz 1)**.** *For each* $n \geq 1$, $\mathcal{B}_n$ *has* $n-1$ *generators* $s_1, \ldots, s_{n-1}$ *satisfying the* ***braid relations***

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

for all $1 \leq i < n-1$, *and*

$$s_i s_j = s_j s_i$$

where $|i-j| > 1$.

Example 8. By Artin's theorem, we can immediately compute

$$\mathcal{B}_1 = \langle \varnothing \,|\, \varnothing \rangle \simeq 0 \quad \text{and} \quad \mathcal{B}_2 = \langle s \,|\, \varnothing \rangle \simeq \mathbb{Z}.$$

However, $\mathcal{B}_3 = \langle s, t \,|\, sts = tst \rangle$ is already much less familiar. In fact, B_3 arises naturally in the context of group extensions of the **modular group** $\mathrm{PSL}(2, \mathbb{Z})$. The modular group is given by the following group presentation

$$\mathrm{PSL}(2, \mathbb{Z}) = \langle A, B \,|\, A^2 = B^3 = 1 \rangle.$$

Notably, a similar relation obtains in $\mathcal{B}_3$: Letting $a = sts$ and $b = st$, observe that

$$a^2 = (sts)(sts) = st(ssts) = st(stst) = (st)^3 = b^3.$$

Conversely, the braid relations can be recovered from the relation $a^2 = b^3$ since

$$tst = (a^{-1}b^2)(b^{-1}a)(a^{-1}b^2) = a^{-1}bb^2 = a^{-1}b^3 = a = sts$$

and so we can write

$$\mathcal{B}_3 = \langle a, b \,|\, a^2 = b^3 \rangle.$$

Hence, we can relate these groups via

$$\mathcal{B}_3 \Big/ (a^2) \simeq \mathrm{PSL}(2, \mathbb{Z}).$$

Since $\mathrm{PSL}(2, \mathbb{Z})$ has a trivial centre, $Z(\mathcal{B}_3) \leq (a^2)$. Moreover, since

$$sa^2 = s(stssts) = (ssts)(sts) = (ststst)s = a^2 s$$

and similarly $ta^2 = a^2 t$, we have $Z(\mathcal{B}_3) = (a^2)$. This implies that $\mathcal{B}_3$ is a **central extension** of the modular group, that is, the sequence

$$1 \longrightarrow Z(\mathcal{B}_3) \longrightarrow \mathcal{B}_3 \longrightarrow \mathrm{PSL}(2, \mathbb{Z}) \longrightarrow 1$$

is exact.

2.2. **The symmetric group S_n.** In this section, we relate the braid groups to the more familiar symmetric groups. The symmetric group S_n, where $n \geq 1$, is the group of permutations of n objects. This group is generated by the adjacent transpositions $(i, i+1)$ for all $1 \leq i \leq n-1$, denoting the permutation swapping the ith and $(i+1)$th objects, while fixing the others. As such, we can represent each permutation as a strand diagram with crossings marked by adjacent transpositions. For example the permutation $\sigma = (1,2,3)(4,5)$ in S_5 can be represented by

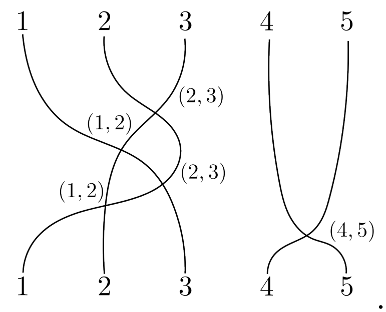


FIGURE 14. Strand representation of $\sigma = (1,2,3)(4,5)$.

Reading from bottom to top, the above strand diagram gives a decomposition of σ into a product of adjacent transpositions, that is, $(1,2,3)(4,5) = (2,3)(1,2)(2,3)(1,2)(4,5)$. Strand diagrams are an easy way to check whether two permutations, written as a product of adjacent transpositions, are really the same.

Letting $s_i := (i, i+1)$ for all $1 \leq i < n$, Figure 15 illustrates some important relations of the symmetric group

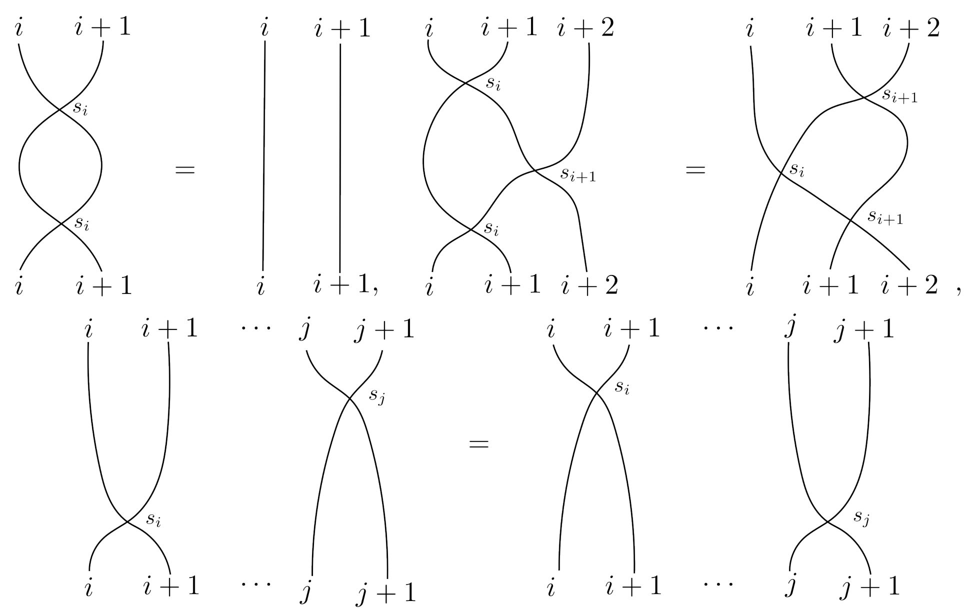


FIGURE 15. Relations of the symmetric group.

In 1896, Eliakim Hastings Moore showed that these relations define the symmetric group S_n on $n-1$ generators.

Theorem 9 (Moore [23], Theorem A). *For each $n \geq 1$, S_n has $n-1$ generators $s_1, \dots, s_{n-1}$ satisfying the **involution relations***

$$s_i^2 = 1$$

*for all $1 \leq i < n$, and the **braid relations***

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

for all $1 \leq i < n-1$, and

$$s_i s_j = s_j s_i$$

where $|i-j| > 1$.

By direct comparison with Theorem 7, the group presentation of $\mathcal{B}_n$ resembles that of the symmetric group S_n. In fact, S_n has the same number of generators and satisfies the braid relations. However, only the symmetric group satisfies the involution relations. As such, $\mathcal{B}_n$ is a group extension of S_n

$$1 \longrightarrow N \longrightarrow \mathcal{B}_n \xrightarrow{\pi} S_n \longrightarrow 1$$

where N is the normal closure of the group generated by the involution relations. We call the surjection $\pi : \mathcal{B}_n \twoheadrightarrow S_n$ the canonical projection map, whose kernel N defines an important subgroup $P\mathcal{B}_n$ called the **pure braid group** on n strands. Geometrically, pure braids are isotopy classes of geometric braids whose strands join the lower points with the points directly above. Figure 16 is an example of a pure braid.

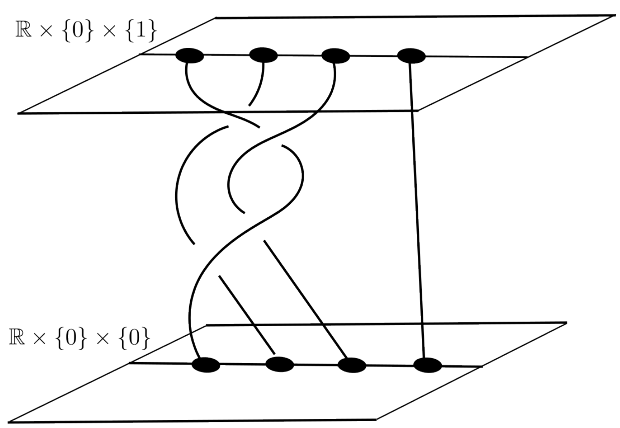


FIGURE 16. Example of a pure braid.

We note that S_n naturally embeds in S_{n+1} as the subgroup of permutations whose $n+1$ object is fixed. Similarly, $\mathcal{B}_n$ embeds in $\mathcal{B}_{n+1}$ as the subgroup of braids whose $n+1$ strand joins $(n+1, 0, 0)$ to $(n+1, 0, 1)$ and is isotopic to the strictly vertical strand. The following diagram therefore commutes

$$\begin{tikzcd} \mathcal{B}_n \arrow[r, two heads] \arrow[d, hook] & S_n \arrow[d, hook] \\ \mathcal{B}_{n+1} \arrow[r, two heads] & S_{n+1}. \end{tikzcd}$$

As an immediate corollary, we have the following proposition.

Corollary 10. *For $n \geq 3$, $\mathcal{B}_n$ is non-abelian.*

Proof. By the diagram above, it suffices to show that $\mathcal{B}_3$ is non-abelian. If it were, then S_3 would be abelian. But S_3 is not abelian. □

2.3. **The configuration space $\mathcal{C}_n(M)$.** The developing theme of this section is that the braid groups are related to the fundamental group of a so called configuration space. This was an observation originally made by Hurwitz in 1891 but rediscovered in 1962 by Fox and Neuwirth. In this subsection, we introduce the configuration space $\mathcal{C}_n(M)$ of a topological manifold M.

Given a topological manifold M we define the **ordered configuration space of n points in** M to be the topological manifold

$$\mathcal{C}_n(M) := M^n - \bigcup_{i \neq j} \{(u_1, \ldots, u_n) \in M^n : u_i = u_j\}.$$

That is, $\mathcal{C}_n(M)$ is the largest subspace of M^n on which the symmetric group S_n acts freely by permuting coordinates. As a consequence, the **unordered configuration space of n points in $\mathbb{C}$**, $\mathcal{C}_n(M)/S_n$ has $\mathcal{C}_n(M)$ as a covering space. Configuration spaces appear naturally in various areas of mathematics and physics, such as in the deformation theory of rational singularities [6].

If the fundamental group of $\mathcal{C}_n(M)$ is to be well-defined, then $\mathcal{C}_n(M)$ must be path-connected. The follow proposition ensures that $\mathcal{C}_n(\mathbb{C})$ is a connected manifold, and therefore path-connected.

Proposition 11. *Let M be a connected topological manifold of $\dim M \geq 2$. Then $\mathcal{C}_n(M)$ is a connected submanifold.*

Proof. Let

$$x := (x_1, \ldots, x_n) \quad \text{and} \quad y := (y_1, \ldots, y_n)$$

be distinct points in $\mathcal{C}_n(M)$. We want to find a path that joins these points. We first consider the case that

$$\{x_1, \ldots, x_n\} \cap \{y_1, \ldots, y_n\} = \varnothing.$$

We define the n-tuples

$$z_k := (y_1, \ldots, y_k, x_{k+1}, \ldots, x_n)$$

for $0 \leq k \leq n$, which are all elements in $\mathcal{C}_n(M)$. Since M is a 2-dimensional connected manifold, removing a finite set of point does not disconnect the space. Therefore, we know that the submanifold $M \backslash \{x_2, \ldots, x_n\}$ of M is connected, so there exists a path γ_1 :

$I \to M \setminus \{x_2, \dots, x_n\}$ such that $\gamma_1(0) = x_1$ and $\gamma_1(1) = y_1$. For each $t \in I$, we know that $\gamma(t) \notin \{x_2, \dots, x_n\}$ and so $(\gamma_1(t), x_2, \dots, x_n) \in \mathcal{C}_n(M)$ for all t. This induces the path $\widetilde{\gamma}_1 : I \to \mathcal{C}_n(M)$ in $\mathcal{C}_n(M)$ from x to z_1. For the same reasoning, we can construct paths in $\mathcal{C}_n(M)$ from z_k to z_{k+1} for each $k \in \{1, 2, \dots, n-1\}$. But this implies that there is a path in $\mathcal{S}_n(M)$ from $x = z_0$ to $y = z_n$.

We now treat the case where

$$\{x_1, \dots, x_n\} \cap \{y_1, \dots, y_n\} \neq \varnothing.$$

We can pick $w := (w_1, \dots, w_n)$ in $\mathcal{C}_n(M)$ such that

$$\{w_1, \dots, w_n\} \cap \{x_1, \dots, x_n, y_1, \dots, y_n\} = \varnothing.$$

Then by applying the above argument to x and w, and then to w and y, we can construct paths in $\mathcal{C}_n(M)$ from x to w and from w to y. Thus $\mathcal{C}_n(M)$ is connected. □

2.4. **Computing $\pi_1(\mathcal{C}_n(\mathbb{C}))$.** We now outline the argument of Fox and Neuwirth, which relates the braid groups to the fundamental group of the configuration spaces of $\mathbb{C}$.

Theorem 12 (Fox and Neuwirth [14], Main Theorem)**.**

$$P\mathcal{B}_n \simeq \pi_1\left(\mathcal{C}_n(\mathbb{C})\right).$$

Proof sketch. The goal is to make precise the geometric relationship between loops in $\mathcal{C}_n(\mathbb{C})$ and pure geometric braids in $\mathbb{R}^2 \times I \simeq \mathbb{C} \times I$. We refer to Figure 17 as a visual aide. Given a loop $f : I \to \mathcal{C}_n(\mathbb{C})$ at $p := (1 + 0i, \dots, n + 0i)$, we obtain a family of continuous functions $u_i : I \to \mathbb{C}$ for $1 \leq k \leq n$ such that $f(t) = (u_1(t), \dots, u_n(t))$ and

$$u_k(0) = u_k(1) = k + 0i \quad \text{for all } 1 \leq k \leq n \quad \text{(loop condition)}$$

and

$$u_k(t) = u_\ell(t) \text{ for some } t \in I \implies k = \ell \quad \text{(distinct points condition).}$$

Using these maps, consider the subregion of $\mathbb{C} \times I$ given by

$$b_f := \bigcup_{k=1}^{n} \bigcup_{t \in I} (u_k(t), t)$$

joining the points $\{(1 + 0i, 0), \dots, (n + 0i, 0)\}$ with the points $\{(1 + 0i, 1), \dots (n + 0i, 1)\}$. The distinct points condition implies that b_f is a geometric braid, since it implies that each strand $(u_k(I), I)$ is disjoint. Furthermore, the loop condition implies that b_f is a pure geometric braid.

Observe that if g is homotopic to f as a loop in $\mathcal{C}_n(\mathbb{C})$, then b_g is isotopic to b_f as a geometric braid in $\mathbb{C} \times I$. Let β_f denote the isotopy class of b_f, then the map $f \mapsto \beta_f$ factors uniquely through $\pi_1(\mathcal{C}_n(\mathbb{C}))$ inducing a well-defined homomorphism $\pi_1(\mathcal{C}_n(\mathbb{C})) \to P\mathcal{B}_n$. A short verification can show that this map is, in fact, an isomorphism.

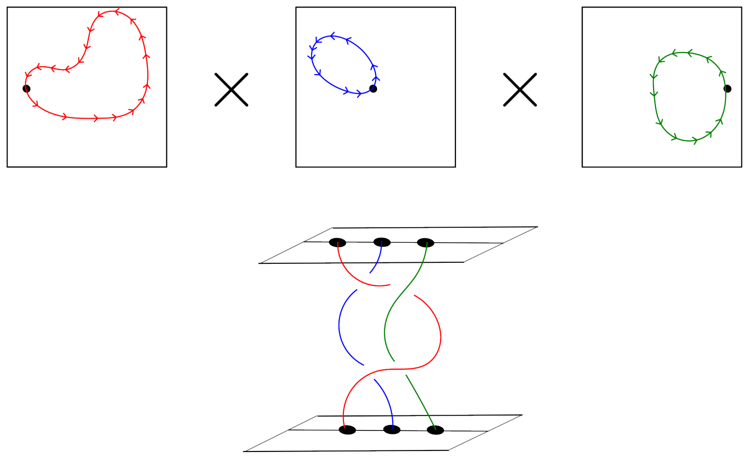


FIGURE 17. An illustration of the relationship between $P\mathcal{B}_3$ and $\mathcal{C}_3(\mathbb{C})$.

$\square$

Let $p : \mathcal{C}_n(\mathbb{C}) \to \mathcal{C}_n(\mathbb{C})/S_n$ be the covering space map. Since S_n is the group of deck transformations of the covering space, we obtain a short exact sequence

$$1 \longrightarrow p_*(\pi_1(\mathcal{C}_n(\mathbb{C}))) \longrightarrow \pi_1\left(\mathcal{C}_n(\mathbb{C})\,/\,S_n\right) \longrightarrow S_n \longrightarrow 1$$

where p_* is the induced homomorphism. A consequence of the homotopy lifting property is that the p_* is injective, and so $p_*(\pi_1(\mathcal{C}_n(\mathbb{C}))) \simeq \pi_1(\mathcal{C}_n(\mathbb{C}))$. This gives us the following corollary.

Corollary 13 (Fox and Neuwirth [14], Main Theorem)**.**

$$\mathcal{B}_n \simeq \mathcal{C}_n(\mathbb{C})\,/\,S_n.$$

Proof. Note that

$$\mathcal{C}_n(\mathbb{C})\,/\,S_n \simeq \left\{\{x_1 + y_1 i, \ldots, x_n + y_n i\} : (x_k, y_k) = (x_\ell, y_\ell) \implies k = \ell\right\}$$

Given a loop $f : I \to \mathcal{C}_n(\mathbb{C})/S_n$ at $p' := \{1 + 0i, \ldots, n + 0i\}$ there exists a family of continuous maps $u_k : I \to \mathbb{C}$ for $1 \leq k \leq n$ such that $f(t) = \{u_1(t), \ldots, u_n(t)\}$ and

$$u_k(t) = u_\ell(t) \text{ for some } t \in I \implies k = \ell \quad \text{(distinct points condition)}.$$

As in Theorem 12, the distinct points condition implies that b_f is a geometric braid in $\mathbb{C} \times I$, however, in this case, the loop condition is weaker, allowing $u_k(1) = u_\ell(0)$ despite $k \neq \ell$ and thus b_f need not be pure. Hence, the map $f \mapsto b_f$ induces a well-defined homomorphism $\pi_1(\mathcal{C}_n(\mathbb{C})/S_n) \to \mathcal{B}_n$. As such, we obtain two rows of exact sequences of the form

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\mathcal{C}_n(\mathbb{C})) \arrow[r] \arrow[d, "\simeq \text{ (Thm 12)}"] & \pi_1\left(\mathcal{C}_n(\mathbb{C}) \big/ S_n\right) \arrow[r] \arrow[d] & S_n \arrow[r] \arrow[d, "\simeq"] & 1 \\
1 \arrow[r] & P\mathcal{B}_n \arrow[r] & \mathcal{B}_n \arrow[r] & S_n \arrow[r] & 1.
\end{tikzcd}$$

By the five lemma from homological algebra, it follows that

$$\mathcal{B}_n \simeq \pi_1\left(\mathcal{C}_n(\mathbb{C}) \big/ S_n\right).$$

□

2.5. **Computing $\pi_i(\mathcal{C}_n(\mathbb{C}))$ for $i \geq 2$.** In this section, we outline the argument of Fadell and Neurwith that the configuration spaces $\mathcal{C}_n(\mathbb{C})$ have trivial higher homotopy groups.

We first consider the fibre structure of $\mathcal{C}_n(\mathbb{C})$. Given the projection $\pi : \mathcal{C}_n(\mathbb{C}) \to \mathbb{C}$ where $\pi(u_1, \ldots, u_n) = u_1$, observe that

$$\pi^{-1}\{\lambda\} \simeq (\mathbb{C} - \{\lambda\})^{n-1} - \bigcup_{i \neq j} H_{ij} \tag{2.1}$$

for any $\lambda \in \mathbb{C}$, where H_{ij} is the subregion in $(\mathbb{C} - \{\lambda\})^{n-1}$ given by the equation $u_i = u_j$. We generalise (2.1) in the following way. Let M be a connected manifold of dimension ≥ 2, then we define the following connected submanifold

$$F_{r,n}(M) := (M - Q_r)^n - \bigcup_{i \neq j} \widetilde{H}_{ij}$$

where $r \geq 0$, $n \geq 1$, Q_r denotes a set of r elements in M and $\widetilde{H}_{ij}$ denotes the region in $(M - Q_r)^n$ defined by $u_i = u_j$. Then, $\mathcal{C}_n(\mathbb{C}) = F_{0,n}(\mathbb{C})$.

If Q_r and Q_r' are distinct sets of r elements in a connected manifold M of dimension ≥ 2, then a standard result in manifold theory is that $M - Q_r$ is homeomorphic to $M - Q_r'$. Hence, $\pi^{-1}(\lambda) \simeq F_{1,n-1}(\mathbb{C})$. The argument depends on the fibre structure of the more general space $F_{r,n}(M)$.

Lemma 14 (Fadell and Neuwirth [13], Theorem 1)**.** *Given the projection $p : F_{r,n}(M) \to M - Q_r$ where $p(u_1, \ldots, u_n) = u_1$, if $n > 1$ then p is a fibre bundle projection with fibres $F_{r+1,n-1}(M)$.*

Theorem 15 (Fadell and Neuwirth [13], Corollary 2.2)**.**

$$\pi_i(\mathcal{C}_n(\mathbb{C})) \simeq 0 \quad i \geq 2.$$

Proof. Since fibre bundles give rise to a long exact sequence of homotopy groups, Lemma 14 implies that the sequence

$$\pi_{i+1}(\mathbb{C} - Q_r) \longrightarrow \pi_i(F_{r+1,n-1}(\mathbb{C})) \longrightarrow \pi_i(F_{r,n}(\mathbb{C})) \longrightarrow \pi_i(\mathbb{C} - Q_r)$$

is exact, provided $n > 1$. The space $\mathbb{C} - Q_r$ contracts to the wedge product of r one-spheres, which has a contractible universal covering space (see Hatcher [15], Example

1.45). Therefore, $\pi_i(\mathbb{C} - Q_r) \simeq 0$ for all $i \geq 2$ and so by exactness

$$\pi_i(F_{r,n}(\mathbb{C})) \simeq \pi_i(F_{r+1,n-1}(\mathbb{C})) \quad i \geq 2.$$

Hence, proceeding inductively

$$\pi_i(\mathcal{C}_n(\mathbb{C})) = \pi_i(F_{0,n}(\mathbb{C})) \simeq \pi_i(F_{n-1,1}(\mathbb{C})) \simeq \pi_i(\mathbb{C} - Q_{n-1}) \simeq 0$$

for all $i \geq 2$. □

From algebraic topology, a covering space projection $p : \mathcal{C}_n(\mathbb{C}) \to \mathcal{C}_n(\mathbb{C})/S_n$ induces an isomorphism $p_* : \pi_i(\mathcal{C}_n(\mathbb{C})) \to \pi_i(\mathcal{C}_n(\mathbb{C})/S_n)$ for all $i \geq 2$. Hence, we get the following corollary for free.

Corollary 16.

$$\pi_i\left(\mathcal{C}_n(\mathbb{C}) / S_n\right) \simeq 0$$

for all $i \geq 2$*.*

2.6. **The group cohomology of $\mathcal{B}_n$.** In this section we outline the cohomological upshot of the results established in §2.4-2.5. We then show that this implies the braid groups are torsion free.

Let X be a path-connected topological space and let G be a group. We say that X is $K(G,1)$ (if G is discrete, we may also say X is a **classifying space** of G) if

$$\pi_1(X) \simeq G \quad \text{and} \quad \pi_i(X) \simeq 0 \text{ for all } i \geq 2.$$

Theorems 12 and 15 tell us that $\mathcal{C}_n(\mathbb{C})/S_n$ is $K(\mathcal{B}_n, 1)$. From algebraic topology, this means the group cohomology of $\mathcal{B}_n$ is precisely the singular cohomology of $\mathcal{C}_n(\mathbb{C})/S_n$:

$$H^i_{\mathrm{Sing}}(\mathcal{C}_n(\mathbb{C})/S_n, \mathbb{Z}) \simeq H^i_{\mathrm{Grp}}(\mathcal{B}_n, \mathbb{Z})$$

for all $i \geq 0$. Since $\mathcal{C}_n(\mathbb{C})/S_n$ is a manifold, this implies that the *braid groups have the cohomology of a manifold*, a type of space whose cohomology is well-understood. The following theorem is a concrete application of this fact.

Theorem 17. *The braid groups are torsion free.*

Proof. For the sake of contradiction, assume that there exists a nonzero finite cyclic subgroup C of $\mathcal{B}_n$. By the Galois correspondence in algebraic topology, there exists a covering space $p : X \to \mathcal{C}_n(\mathbb{C})$ such that $\pi_1(X) \simeq C$. Moreover, since covering spaces induce isomorphisms of the form $\pi_i(X) \simeq \pi_i(\mathcal{C}_n(\mathbb{C})) \simeq 0$ for $i \geq 2$, it follows that X is $K(C,1)$. As such, for all $i \geq 0$

$$H^i_{\mathrm{Sing}}(X, \mathbb{Z}) \simeq H^i_{\mathrm{Grp}}(C, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ C & \text{if } i \text{ is odd} \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Since $\mathcal{C}_n(\mathbb{C})$ is a $2n$-dimensional manifold, so too is X. However, a familiar result in the cohomology theory of manifolds is that $H^i_{\mathrm{Sing}}(X, \mathbb{Z}) \simeq 0$ for all $i > 2n$. This contradicts (2.2). □

3. Coxeter groups

In Chapter 2 we discovered the remarkable fact that the configuration space $\mathcal{C}_n(\mathbb{C})/S_n$ is $K(\mathcal{B}_n, 1)$. The French mathematician Jacques Tits conjectured that this fact held for a more general class of objects called the Artin braid groups [6]. In this chapter our ultimate goal is to construct the Artin braid groups, but to do this we must develop the theory of Coxeter groups which themselves generalise reflection groups.

We begin by introducing Coxeter systems, list several examples and study some of the combinatorial features of Coxeter groups (§3.1-3.2). We then introduce the reflection representation of a Coxeter system, on which we may construct a root system, allowing us to transform combinatorial features of our Coxeter groups into combinatorial features about roots in a vector space (§3.3-3.4). In §3.5 we establish a correspondence between the positive roots of a finite Coxeter group and the hyperplanes associated to that group. Given its importance in the theory, we then take a detour in §3.6 to sketch out the proof of the classification of finite Coxeter groups. In the final section (§3.7) we define the Artin braid group B_W associated to the Coxeter system (W, S).

For a more detailed treatment of Coxeter groups, see Humphreys' text "Reflection groups and Coxeter groups" [18]. And for more on Artin braid groups see [7].

3.1. Coxeter systems (W, S).

Let S be a non-empty finite set. We define a **Coxeter group** to be a group given by a **Coxeter presentation**

$$W := \langle S \mid I \sqcup B \rangle$$

where $I := \{s^2 : s \in S\}$ is a fixed set of involution relations and B is a set of **Artin braid relations** that take the form $(st)^{m(s,t)}$ where $m(s,t) \in \{2, 3, 4, \ldots\}$ and $s \neq t$. Note, due to the involution relations, $m(s,t) = m(t,s)$. Conventionally, we set $m(s,t) := \infty$ if a relation between s and t is not defined. We say that $|S|$ is the **rank** of W.

In general, a Coxeter group need not have a unique Coxeter presentation. In fact, some Coxeter groups can be written in terms of Coxeter presentations with distinct rank. To avoid any confusion, we specify the generating set S of the group W, that is, we will work with the **Coxeter system** (W, S). To specify a Coxeter system is to state a $|S| \times |S|$ symmetric matrix M, with entries corresponding to $m(s,t) \in \{1, 2, 3, \ldots\} \cup \{\infty\}$ subject to the restrictions $m(s,s) = 1$ and $m(s,t) \neq 1$ if $s \neq t$. Such a matrix can be efficiently encoded in a **Coxeter graph** Γ:

> Let S be the vertex set. If $m(s,t) \geq 3$, join an undirected edge between the vertices s and t; if additionally, $m(s,t) \geq 4$, label the edge with the value $m(s,t)$.

Note, if there is an unlabelled edge between s and t then we can read off $m(s,t) = 3$. Moreover, if $s \neq t$ and there is no edge between s and t, we can read off that $m(s,t) = 2$. There are no loops in Γ since $m(s,s) < 3$. We will denote the Coxeter group specified by a Coxeter graph Γ by W_Γ.

Example 18. For $n \geq 3$, the symmetries of an n-gon form a group under composition called the **dihedral group** $I_2(n)$ of order $2n$. The elements of this group consist of n reflections and n rotations. In fact, the n rotations arise by composing two reflections whose axes meet at an angle of π/n, as illustrated by Figure 18.

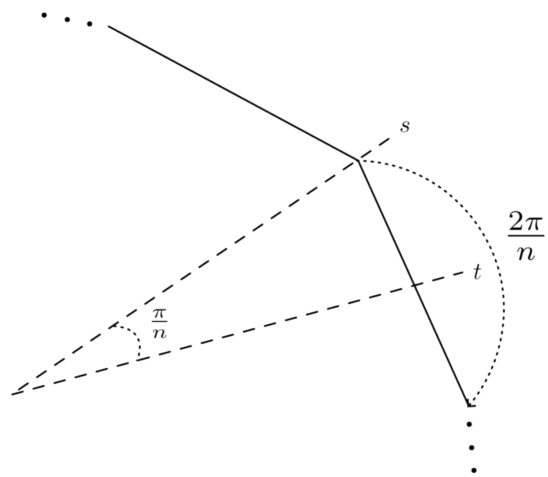
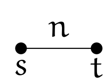


FIGURE 18. Rotation by π/n is obtained by composing reflections s and t.

Let s and t denote the reflections through the axes pictured above. Then the rotations are $1, st, (st)^2, \ldots, (st)^{n-1}$ and the reflections are $s, t, tst, \ldots, t(st)^{n-2}$. In other words, $I_2(n)$ is a finite Coxeter group specified by the Coxeter graph



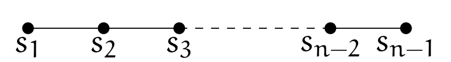
or the Coxeter matrix

$$\begin{pmatrix} 1 & n \\ n & 1 \end{pmatrix}.$$

Example 19. The symmetric group S_n is a Coxeter group. From Theorem 9, we know that S_n is generated by $n-1$ elements $s_1, \ldots, s_{n-1}$ with relations

$$s_i^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i$$

for all i and $|j-i| > 1$. These relations are equivalent to the Coxeter relations $(s_i s_j)^{m(i,j)}$ where $m(i,i) = 1$, $m(i,i+1) = 3$ and $m(i,j) = 2$ when $|j-i| > 1$. Thus, S_n is a Coxeter group specified by the Coxeter graph



or Coxeter matrix

$$\overbrace{\begin{pmatrix} 1 & 3 & 2 & \cdots & \cdots & 2 \\ 3 & 1 & 3 & \cdots & \cdots & 2 \\ 2 & 3 & 1 & \cdots & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 2 & 2 & 2 & \cdots & \cdots & 1 \end{pmatrix}}^{(n-1)\times(n-1)}.$$

Example 20. In a vector space V, call the subspaces U with $\operatorname{codim} U = 1$ the **hyperplanes** of V. In Example 18, we observed that $I_2(n)$ is generated by the reflections s and t through two hyperplanes H_s and H_t in $\mathbb{R}^2$ that meet at an angle of π/n. Therefore, as $n \to \infty$, the angle between the hyperplanes H_s and H_t tends to zero while the order of the rotation st tends to infinity. Thus, in the limit $n \to \infty$, we may regard the hyperplanes as parallel lines in the upper half-plane of $\mathbb{R}^2$ (Figure 19).

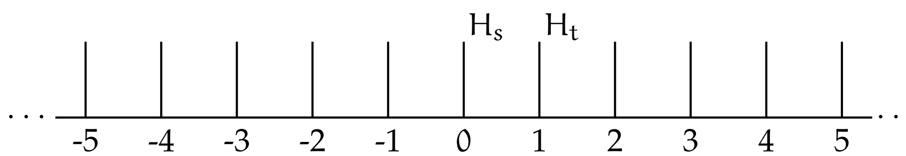


FIGURE 19. Infinite dihedral group generated by reflections through parallel lines.

Each reflection through one of the vertical lines $x = a$ (pictured above) can be regarded as a reflection through the hyperplane defined by $x = 0$ followed by a horizontal translation of $2a$. We call this group the **infinite dihedral group** $I_2(\infty)$. This is an infinite group given by the Coxeter graph

$$\bullet\overset{\infty}{\text{———}}\bullet$$

and Coxeter matrix

$$\begin{pmatrix} 1 & \infty \\ \infty & 1 \end{pmatrix}.$$

Example 21. Let M be a 2-dimensional Riemannian manifold (M, g) with constant Gauss curvature whose geodesics admit reflection isometries (see, for example, symmetric spaces). Given a geodesic triangle T with sides s_a, s_b and s_c in M, we have the associated reflections, called *mirrors*, (a, b, c) that fix a respective side of the triangle and flips the space through it. If ℓ, m and n are integers greater than 1, we call the group generated by mirrors of a triangle the **triangle group** $\Delta(\ell, m, n)$ where π/ℓ, π/m, and π/n are the interior angles of the triangle. The mirrors can be shown to satisfy the following properties

$$a^2 = b^2 = c^2 = 1 \quad \text{and} \quad (ab)^\ell = (bc)^m = (ac)^n = 1$$

where, for instance, the sides corresponding to a and b meet at the angle π/ℓ, and so on. Each side of a triangle, can therefore reflect the other sides through it, forming an adjacent congruent triangle in the space (Figure 20).

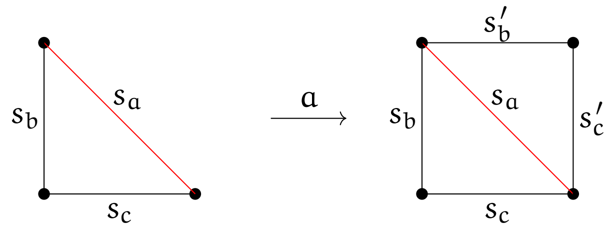


FIGURE 20. Generating a congruent triangle via the mirror a in $\mathbb{R}^2$ with Euclidean metric.

The images of triangles generated by these mirrors forms a tiling of M (see, for example, Figure 21).

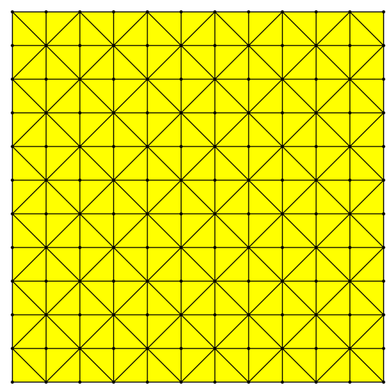


FIGURE 21. Tetrakis tilling of $\mathbb{R}^2$ [29].

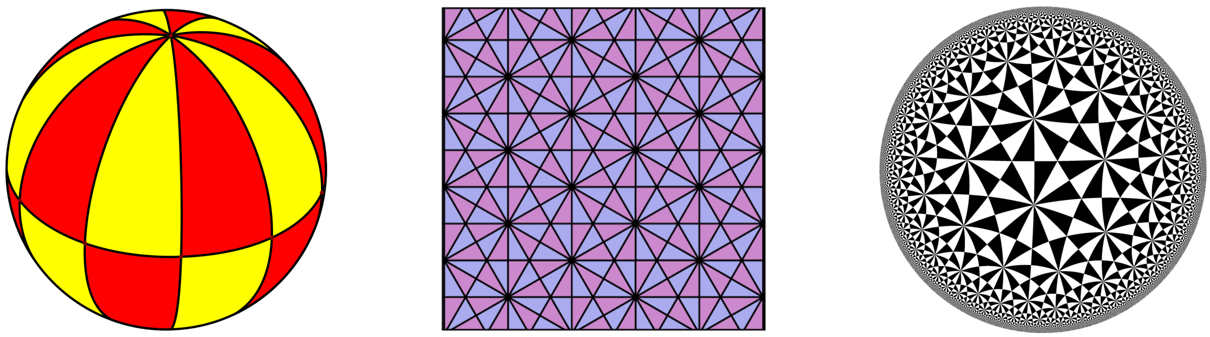
There is classical result in differential geometry due to Gauss (1827) which implies the following.

> Let T be a geodesic triangle, with unit area, of a Riemannian surface with constant Gauss curvature K. Then
> $$\pi/\ell + \pi/m + \pi/n = \pi + K.$$

When the hypotheses of the theorem hold, we can say something about the curvature of a 2 dimensional symmetric space M by summing the interior angles of a geodesic triangle; namely,

$$\begin{aligned}
\pi/\ell + \pi/m + \pi/n > \pi &\iff K > 0 \iff M \text{ is spherical}\\
\pi/\ell + \pi/m + \pi/n = \pi &\iff K = 0 \iff M \text{ is euclidean}\\
\pi/\ell + \pi/m + \pi/n < \pi &\iff K < 0 \iff M \text{ is hyperbolic.}
\end{aligned}$$

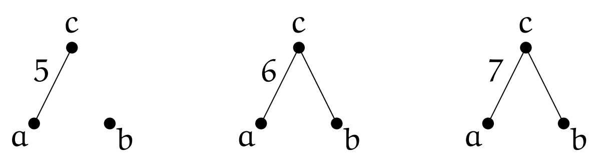
As such, for any triangle group $\Delta(\ell, m, n)$, it can be regarded as the group generated by the mirrors of a triangle in a sphere, or 2-dimensional Euclidean or hyperbolic space. We denote Δ_S, Δ_E and Δ_H to denote a triangle group of the respective geometry.



(A) $(\ell, m, n) = (2, 2, 5)$ (B) $(\ell, m, n) = (2, 3, 6)$ (C) $(\ell, m, n) = (2, 3, 7)$

FIGURE 22. Tiling of sphere, euclidean space and Poincare model of hyperbolic space generated by a geodesic triangle [30, 24].

The above triangle groups are given, respectively, by the following Coxeter graphs



or, by the Coxeter matrices

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 1 & 2 \\ 5 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 7 \\ 2 & 1 & 3 \\ 7 & 3 & 1 \end{pmatrix}.$$

3.2. **Reduced expressions.** In this section we define a length function $\ell : W \to \mathbb{N}$ on W. This provides a tool for studying Coxeter groups via induction on the lengths of its elements. We also introduce parabolic subgroups.

By the involution relations of W, any element w in W can be expressed as a finite product of the form $s_1 s_2 \cdots s_r$ such that $s_i \neq s_{i+1}$ for all $1 \leq i < r$. However, the index r need not be unique. For example, if $W = S_3$, then the two expressions $s_1 s_2$ and $s_2 s_1 s_2 s_1$ are equivalent. If $s_1 \cdots s_r$ is an expression as above such that for any other equivalent expression $t_1 \cdots t_{r'}$, we have $r \leq r'$, then we say $s_1 \cdots s_r$ is a **reduced expression**. We can therefore define a **length function** $\ell : W \to \mathbb{N}$ given by $\ell(w) = r$ where r is the number of generators in a reduced expression of w.

In general, the length of ww' where w and w' are elements of W cannot be determined by the lengths of w and w' individually. However, we can obtain some elementary properties about the length function.

Proposition 22. *The lenth function satisfies the following properties:*

(L1) $\ell(w^{-1}) = \ell(w)$;
(L2) $\ell(ww') \leq \ell(w) + \ell(w')$;
(L3) $\ell(ws) = \ell(w) + 1$ *or* $\ell(ws) = \ell(w) - 1$

where $w, w' \in W$ *and* $s \in S$.

Proof. Let $s_1 \cdots s_r$ be a reduced expression of w.

(L1): Observe that $(s_1 \cdots s_r)^{-1} = s_r^{-1} \cdots s_1^{-1} = s_r \cdots s_1$. If $s_r \cdots s_1$ is not reduced, then there exists $r' < r$ such that $w^{-1} = t_1 \cdots t_{r'}$ is reduced. But then $w = (w^{-1})^{-1} = t_{r'} \cdots t_1$ is a reduced expression of w where $r' < r$, which is a contradiction. So $\ell(w^{-1}) = r = \ell(w)$.

(L2): If $t_1 \cdots t_s$ is a reduced expression of w', then $s_1 \cdots s_r t_1 \cdots t_s$ is an expression of ww'; hence $\ell(ww') \leq r + s = \ell(w) + \ell(w')$.

(L3): Firstly, (L2) implies that

$$\ell(ws) \leq \ell(w) + \ell(s) = \ell(w) + 1. \tag{3.1}$$

Moreover,

$$\ell(w) = \ell(ws^2) \leq \ell(ws) + \ell(s) \implies \ell(ws) \geq \ell(w) - \ell(s) = \ell(w) - 1. \tag{3.2}$$

If $s_1 \cdots s_r s$ is reduced, then

$$\ell(ws) > \ell(w) \overset{(3.1)}{\implies} \ell(ws) = \ell(w) + 1.$$

If $s_1 \cdots s_r s$ is not reduced, then $\ell(ws) \leq \ell(w)$. Note, if the equality holds, then there is a reduced expression $ws = t_1 \cdots t_r$ and so $t_1 \cdots t_r s$ and $s_1 \cdots s_r$ are both expressions of w. Consider the map $S \to \mathbb{Z}/2\mathbb{Z}$ sending $s \mapsto 1$. This map extends to a well-defined group homomorphism $\varepsilon : W \to \mathbb{Z}/2\mathbb{Z}$ since every relation in W is of the form $(st)^m = 1$, and so $\varepsilon((st)^m) = 2m = 0$. Observe that

$$\varepsilon(w) = \varepsilon(s_1 \cdots s_r) = r \quad \text{and} \quad \varepsilon(w) = \varepsilon(t_1 \cdots t_r s) = r + 1.$$

But this is a contradiction since $r \neq r + 1 \mod 2$. Hence

$$\ell(ws) < \ell(w) \overset{(3.2)}{\implies} \ell(ws) = \ell(w) - 1.$$

□

Let $I \subseteq S$. There is a natural group homomorphism from $f : F(I) \to W$ where $F(I)$ is the free group generated by I. We define $W_I := f(F(I))$ to be a **parabolic subgroup** of W. In particular, if $|I| = 2$, we say that W is a **dihedral subgroup** of W_I. Moreover, we can similarly define a length function $\ell_I : W_I \to \mathbb{N}$ given by $\ell_I(w) = r$ where r is the length of the shortest expression of w in the generators I. A priori, $\ell_I(w) \geq \ell(w)$ for all $w \in W_I$. Parabolic subgroups play an important role in the theory of Coxeter groups [18]; in this document their role is auxiliary for the proof of Theorem 24.

3.3. **Reflection representation** V**.** We construct the reflection representation of (W, S). Let V be the real vector space spanned by the basis $\{\alpha_s : s \in S\}$, equipped with the symmetric bilinear form $B : V \times V \to \mathbb{R}$ given by

$$B(\alpha_s, \alpha_t) = -\cos\left(\frac{\pi}{m_{st}}\right), \text{ if } m_{st} < \infty \qquad B(\alpha_s, \alpha_t) = -1, \text{ otherwise.}$$

For each generator s in S, we define the following invertible linear map

$$\sigma_s\lambda = \lambda - 2B(\lambda, \alpha_s)\alpha_s.$$

We make two observations about σ_s. Firstly, given any $\lambda = c\alpha_s$ along the line $\mathbb{R}\alpha_s$,

$$\sigma_s\lambda = c\alpha_s - 2B(c\alpha_s, \alpha_s)\alpha_s = c\alpha_s - 2c\alpha_s = -\lambda.$$

Secondly, for any μ in the orthogonal complement $(\mathbb{R}\alpha_s)^\perp = \{\mu \in V : B(\mu, \alpha_s) = 0\}$, $\sigma_s\mu = \mu - 2B(\mu, \alpha_s)\alpha_s = \mu$. Any invertible linear map that sends a nonzero vector α to $-\alpha$ and fixes the hyperplane $(\mathbb{R}\alpha)^\perp$ is called a **reflection** of V. As such, σ_s for any $s \in S$ is a reflection of V.

Proposition 23. *The maps $s \mapsto \sigma_s$ extend to a representation $\sigma : W \to GL(V)$ called the* ***reflection representation*** *of W.*

Proof. The maps $s \mapsto \sigma_s$ extend to a group homomorphism $f : F(S) \to GL(V)$. The goal of the proof is to apply the universal property of quotients to obtain a group homomorphism $\hat{f} : W \to GL(V)$ so that the diagram

$$\begin{tikzcd} F(S) \arrow[r, "f"] \arrow[d, two heads, "\pi"'] & GL(V) \\ W \arrow[ur, dashed, "\hat{f}"'] & \end{tikzcd}$$

commutes. This is equivalent to showing that

$$(\sigma_s\sigma_t)^{m(s,t)} = \mathrm{id}_V$$

for any $s, t \in S$ whenever $m(s,t) < \infty$. Observe that

$$\sigma_s^2\lambda = \sigma_s(\lambda - 2B(\lambda, \alpha_s)\alpha_s) = \lambda - 4B(\lambda, \alpha_s)\alpha_s + 4B(\lambda, \alpha_s)\overbrace{B(\alpha_s, \alpha_s)}^{=1}\alpha_s = \lambda.$$

We now assume $s \neq t$. Let $m := m(s,t)$ and consider the two-dimensional subspace $V_{s,t} := \mathbb{R}\alpha_s \oplus \mathbb{R}\alpha_t$. Moreover we define B' to be the restriction of B to $V_{s,t}$. We first show that B' is positive semi-definite. Letting $\lambda = c_s\alpha_s + c_t\alpha_t$, we observe that

$$\begin{aligned} B'(\lambda,\lambda) &= a^2\overbrace{B(\alpha_s, \alpha_s)}^{=1} + 2abB(\alpha_s, \alpha_t) + b^2\overbrace{B(\alpha_t, \alpha_t)}^{=1} \\ &= a^2 - 2ab\cos\left(\frac{\pi}{m}\right) + b^2 \\ &= \left(a - b\cos\left(\frac{\pi}{m}\right)\right)^2 + b^2\sin^2\left(\frac{\pi}{m}\right) \geq 0. \end{aligned}$$

Moreover, if $B'(\lambda,\lambda) = 0$, then $b^2\sin^2(\pi/m) = 0$. If $\sin^2(\pi/m) \neq 0$ then $b^2 = 0$ and so $a = 0$. Therefore, if $m < \infty$, B' is positive definite, and so σ_s and σ_t are orthogonal reflections. Moreover, since $B'(\alpha_s, \alpha_t) = -\cos(\pi/m) = \cos(\pi - \pi/m)$ the angle between $\mathbb{R}\alpha_s$ and $\mathbb{R}\alpha_t$ in $V_{s,t}$ is $\pi - \pi/m$. Hence, the angle between the reflecting hyperplanes is

π/m (see Figure 23). Thus, as in Example 18, $\sigma_s\sigma_t$ is a rotation of $V_{s,t}$ by an angle of π/m. Hence $(\sigma_s\sigma_t)^m = \mathrm{id}$.

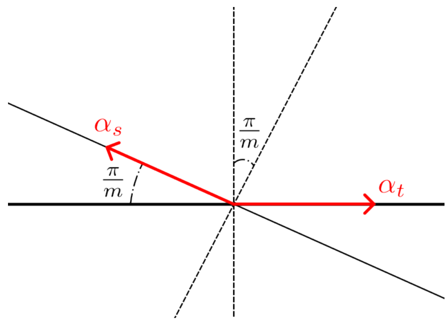


FIGURE 23. $V_{s,t}$ when $m < \infty$.

□

For convenience, we write $w \cdot \lambda := \sigma(w)\lambda$ for all w in W and λ in V.

3.4. **Root system of W.** In this section we show that the reflection representation of a Coxeter group is faithful. This fact depends on a duality between the behaviour of the length function on reduced expressions of W and positivity features of points in the reflection representation known as roots (Theorem 24).

One can directly verify that $B(w \cdot \lambda, w \cdot \mu) = B(\lambda, \mu)$ for all w in W and λ, μ in V. As such, the set

$$\Phi := \{w \cdot \alpha_s : w \in W \text{ and } s \in S\}$$

is a W-invariant collection of unit vectors in V. Moreover, $\Phi = -\Phi$ since $ws \cdot \alpha_s = -w \cdot \alpha_s$. We say that Φ is the **root system** of W and call each $\alpha := w \cdot \alpha_s$ a **root**, and in particular we call $\{\alpha_s\}_{s \in S}$ the **simple roots**. Given a root α, we may express it as a linear combination of the simple roots $\sum c_s \alpha_s$. We then say that α is **positive** (resp. **negative**) if $c_s \geq 0$ (resp $c_s \leq 0$) for all s in S and write $\alpha > 0$ (resp. $\alpha < 0$).

The following theorem, among providing a criterion for the positivity of roots in Φ, further implies that Φ is partitioned into the set of positive roots Φ^+, and negative roots Φ^-.

Theorem 24. *Let $w \in W$ and $s \in S$. Then*

$$\ell(ws) > \ell(w) \iff w \cdot \alpha_s > 0.$$

Proof. We note that is sufficient to show that

$$\ell(ws) > \ell(w) \implies w \cdot \alpha_s > 0 \tag{3.3}$$

since

$$w \cdot \alpha_s > 0 \implies ws \cdot \alpha_s = -w \cdot \alpha_s \implies \ell(ws) > \ell((ws)s) = \ell(w).$$

We proceed inductively on the length of elements in W. If $\ell(w) = 0$, then $w = 1$ and we see that $\ell(1s) = \ell(s) = 1 > 0 = \ell(1)$ and $1 \cdot \alpha_s > 0$, so the theorem holds trivially. We set up the inductive hypothesis: For all $w \in W$, if $\ell(w) < n$ for some $n \in \mathbb{N}$, then we assume (3.3) holds. Now suppose $\ell(w) = n$. The proof strategy is to break up w into two smaller words that are easier to analyse via the inductive hypothesis. Since $\ell(w) > 0$, there exists a generator s' of W such that $\ell(ws') < \ell(w)$ (for example, in a reduced expression of w, take the right-most generator). Moreover, $s \neq s'$ since the length of ws and ws' differ. Thus, for $I = \{s, s'\}$, W_I is a dihedral subgroup of W. We want to write $w = vv_I$ where $v \in W$ and $v_I \in W_I$ such that $v_I \neq w$ and such that $\ell(w) = \ell(v) + \ell_I(v_I)$. For this to be possible, we need the set

$$\Lambda := \{v \in W : v^{-1}w \in W_I \text{ and } \ell(v) + \ell_I(v^{-1}w) = \ell(w)\}$$

to contain an element that is not w. We know such an element exists, since $(ws')^{-1}w \in W_I$ and $\ell(ws') + \ell_I(s') = \ell(w) - 1 + 1 = \ell(w)$, and so $ws' \in \Lambda$. Thus, let $v \in \Lambda \backslash \{w\}$ and assume v has minimal length with respect to the elements in Λ. Let $v_I := v^{-1}w$, then $w = vv_I$. For the sake of contradiction, suppose $\ell(vs) < \ell(v)$. Then

$$\ell(w) = \ell(vs(vs)^{-1}w) \leq \ell(vs) + \ell_I(sv^{-1}w) = \ell(v) - 1 + \ell_I(sv^{-1}w)$$

since $sv^{-1}w \in W_I$. Hence

$$\ell(v) - 1 + \ell_I(sv^{-1}w) \leq \ell(v) - 1 + \ell_I(v^{-1}w) + 1 = \ell(v) + \ell_I(v^{-1}w) = \ell(w)$$

which forces $vs \in \Lambda$ and so by minimality of v, $\ell(vs) \geq \ell(v)$, contradicting the assumption. Hence $\ell(vs) > \ell(v)$. But since $\ell(v) < \ell(w)$, by the inductive hypothesis it follows that $v \cdot \alpha_s > 0$. Therefore, it suffices to show that $v_I \cdot \alpha_s > 0$ since $w = vv_I$.

Note that $\ell_I(v_I s) \geq \ell_I(v_I)$, since if otherwise

$$\ell(ws) \leq \ell(v) + \ell_I(v_I s) < \ell(v) + \ell_I(v_I) = \ell(w)$$

which contradicts the main assumption of our argument. Hence, any reduced expression of v_I must end in s'; that is, a reduced expression of v_I is an alternating product of s and s' ending in s'. There are two cases:

Case 1: $m(s, s') = \infty$. Then $B(\alpha_s, \alpha_{s'}) = -1$, and so one can show that

$$w \cdot \alpha_s = \begin{cases} n\alpha_s + (n+1)\alpha_{s'} & \text{if } n \text{ is odd} \\ (n+1)\alpha_s + n\alpha_{s'} & \text{if } n \text{ is even.} \end{cases}$$

In either case, $w \cdot \alpha_s > 0$.

Case 2: $m := m(s, s') < \infty$. We first note that $\ell_I(v_I) < m$, since if otherwise, by the relation

$$\overbrace{st \cdots st}^{m \text{ times}} = \overbrace{ts \cdots ts}^{m \text{ times}}$$

there would exist a reduced expression of v_I ending in s. Hence, v_I has a reduced form either as $(st)^k$ or $t(st)^k$ where $2k + 1 < m$. The reflection representation $\sigma : W \to GL(V)$ induces a homomorphism $\sigma_I : W_I \to GL(U)$ via restriction, where $U = \text{span}_{\mathbb{R}}\{\alpha_s, \alpha_t\}$. One can show that B restricted to U is positive definite. As such, the roots α_s and α_t meet at an angle of $\pi - \pi/m$ since

$$B(\alpha_s, \alpha_t) = -\cos\left(\frac{\pi}{m}\right) = \cos\left(\pi - \frac{\pi}{m}\right).$$

We refer to Figure 24 to illustrate the following argument: we note that the action of $(st)^k$ rotates α_s an angle of $2\pi k/m$ in the direction of α_t. Note that we have the upper bound

$$\frac{2\pi k}{m} \leq \frac{\pi(m-2)}{m} = \pi - \frac{2\pi}{m}.$$

This sends α_s to a point contained in the cone spanned by α_s and α_t. In particular, it is contained in the positive span of α_s and the vector $(1, \pi/m)$ written in polar form. Hence an additional reflection by t, also sends α_s to a vector in the cone spanned by α_s and α_t. Thus $v_I \cdot \alpha_s > 0$.

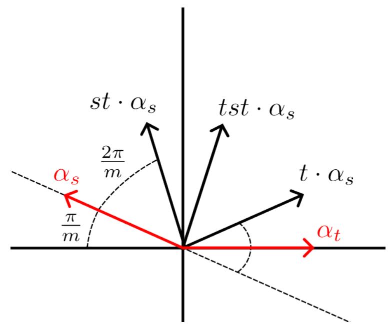


FIGURE 24. The root $v_I \cdot \alpha_s$ is in the cone spanned by α_s and α_t.

□

As an immediate corollary we obtain the following result.

Corollary 25. *The reflection representation $\sigma : W \to GL(V)$ is faithful.*

Proof. Take $w \in \ker \sigma$. For the sake of contradiction, assume that $\ell(w) > 0$. Then $w \neq 1$, and so there exists $s \in S$ such that $\ell(ws) < \ell(w)$. By Theorem 24, $w \cdot \alpha_s < 0$, which implies the contradiction $w \notin \ker \sigma$. Hence $\ell(w) = 0$, and so $w = 1$. □

We can therefore identify $W \simeq \sigma(W)$ as a group generated by reflections of V. In fact, there are many different ways to represent Coxeter groups as groups generated by some notion of a reflection. For instance, one could study groups generated by reflections on the sphere or in hyperbolic space (for more details, see the following papers of Vinberg [31, 32, 33]). We now finish this section with a more precise description of how W permutes the root system Φ in V.

Proposition 26. *Let $s \in S$ and $w \in W$. Then*

(i) $s \cdot \Phi^+ \backslash \{\alpha_s\} = \Phi^+ \backslash \{\alpha_s\}$; *and*
(ii) $\ell(w) = |\{\alpha \in \Phi^+ : w \cdot \alpha < 0\}|$.

Proof. (i): Let $\alpha \in \Phi^+ \backslash \{\alpha_s\}$ and write $\alpha = \sum_{t \in S} c_t \alpha_t$. Since roots are unit vectors, α is not a scalar of α_s, so there exists $t \neq s$ such that $c_t > 0$. Moreover, if $s \cdot \alpha = \sum_{t \in S} c_t' \alpha_s$, then for any $t \neq s$, it follows that

$$s \cdot \alpha_t = \alpha_t - 2B(\alpha_t, \alpha_s)\alpha_s \implies c_t' = c_t > 0.$$

Hence $s \cdot \alpha > 0$ and $s \cdot \alpha \neq \alpha_s$ since there exists $t \neq s$ such that $c_t' = c_t \neq 0$. Hence $s \cdot \Phi^+ \backslash \{\alpha_s\} \subseteq \Phi^+ \backslash \{\alpha_s\}$, and equality follows from applying s to both sides of the set inclusion.

(ii): We define the function $n : W \to \mathbb{N}$ given by $n(w) := |\{\alpha \in \Phi^+ : w \cdot \alpha < 0\}|$. We note the following similarity between ℓ and n. If $w \cdot \alpha_s > 0$, then w only sends roots in $\Phi^+ \backslash \{\alpha_s\}$ to Φ^-. Moreover, by (i) s permutes $\Phi^+ \backslash \{\alpha_s\}$ and $s \cdot \alpha_s = -\alpha_s$ and so

$$\{\alpha \in \Phi^+ : ws \cdot \alpha < 0\} = \{\alpha_s\} \sqcup \{\alpha \in \Phi^+ \backslash \{\alpha_s\} : w \cdot \alpha < 0\}.$$

Hence $n(ws) = n(w) + 1$. Similarly, if $w \cdot \alpha_s < 0$, then $ws \cdot \alpha_s > 0$ and so $n(ws) = n(w) - 1$.

We now show that $n = \ell$ through induction on $\ell(w)$. If $\ell(w) = 0$, then $w = 1$ and so $n(w) = n(1) = 0$. Let $k > 0$ and assume $n(w) = \ell(w)$ for all $\ell(w) < k$. If w has length $\ell(w) = k$, then there exists a reduced expression $w = s_1 \cdots s_{k-1} s_k$. Letting $w' := s_1 \cdots s_k$, by the inductive hypothesis $n(w') = \ell(w')$ and so

$$n(w) = n(w's_k) = \begin{cases} n(w') + 1 & \text{if } w' \cdot \alpha_{s_k} > 0 \\ n(w') - 1 & \text{if } w' \cdot \alpha_{s_k} < 0 \end{cases} \overset{\text{Thm } 24}{=} \ell(w).$$

$\square$

3.5. **Reflections and hyperplanes.** In this section we relate finite Coxeter groups to an arrangement of hyperplanes in V. We start by introducing an important finiteness criterion for Coxeter groups.

Theorem 27. *W is finite if and only if B is positive definite.*

Proof. See Bourbaki [5], Theorem 5.4.8.2. $\square$

The following proposition provides a description of the reflections of W.

Proposition 28. *Let T be the set of elements in W acting as reflections on V. Then*

$$T = \bigcup_{w \in W} wSw^{-1}.$$

We say $w \in T$ is a reflection of W.

Proof. Suppose t is a reflection in W. If $t = s_1 \cdots s_r$ is in reduced form, then $t = s_r \cdots s_1$ is also in reduced form since $t^2 = 1$. If $r = 1$, then $t \in S$ and we are done. Assume inductively that if $r < n$ ($n \geq 2$) then t is a conjugate of a simple reflection. If $r = n$, it follows by the exchange condition (Humphreys [18], Theorem 5.8)

$$\ell(ts_n) = \ell(s_1 \cdots s_{n-1}) < \ell(t) \implies ts_n = s_n \cdots \hat{s_i} \cdots s_1.$$

If $i \neq n$, then $s_n t s_n = s_{n-1} \cdots \hat{s_i} \cdots s_1$ and so $\ell(s_n t s_n) = \ell(t) - 2 < n$. Since $s_n t s_n$ is a reflection, by the inductive hypothesis there is $w \in W$ and $s \in S$ such that $t = (s_n w) s (s_n w)^{-1}$. If $i = n$, then $t s_n = s_{n-1} \cdots s_1$ and so $t = s_n t s_n$. Since $t = s_1 \cdots s_n$ is reduced, the cancellations that occur in the expression $s_n t s_n$ must always involve the left-most or right-most terms (exchange condition), namely s_n. However, since $s_n t = s_{n-1} \cdots s_1$, there must exist some $j \in \{1, \ldots, n-1\}$ such that $s_n t s_n = s_{n-1} \cdots \hat{s_j} \cdots s_1 \hat{s_n}$. But this implies that $\ell(s_n t s_n) = \ell(t) - 2$, contradicting $t = s_n t s_n$. So $i = n$ only when $n = 1$. Therefore every reflection is conjugate to a simple reflection. $\square$

Another way to view the reflections of W is via the roots of Φ. Given a root $\alpha = w \cdot \alpha_s$ in Φ, there is a corresponding reflection

$$\begin{aligned} wsw^{-1} \cdot \lambda &= w \cdot \left(w^{-1} \cdot \lambda - 2B(w^{-1} \cdot \lambda, \alpha_s)\alpha_s \right) \\ &= \lambda - 2B(\lambda, w \cdot \alpha_s)(w \cdot \alpha_s) \\ &= \lambda - 2B(\lambda, \alpha)\alpha \end{aligned}$$

sending α to $-\alpha$. Letting $s_\alpha(\lambda) := \lambda - 2B(\lambda, \alpha)\alpha$, we obtain a surjection $\Phi \twoheadrightarrow T$, $\alpha \mapsto s_\alpha$ since every reflection can be written in the form wsw^{-1} (Proposition 28) and so the root $w \cdot \alpha_s \mapsto s_{w \cdot \alpha_s} = \sigma(wsw^{-1})$. Restricting to the positive roots (alternatively, negative roots), we obtain a bijection

$$\Phi^+ \longleftrightarrow T$$

since, if $s_\alpha = s_\beta$ ($\alpha, \beta \in \Phi^+$) then $s_\alpha(\beta) = s_\beta(\beta)$ and by the positivity of α and β it follows that

$$B(\beta, \alpha)\alpha = B(\beta, \beta)\beta = \beta \implies \beta \in \mathbb{R}^{>0}\alpha \implies \alpha = \beta.$$

Having established a relationship between the root system Φ and the reflections T, there is a third component to consider. Given a root α in Φ, the reflection s_α fixes a hyperplane of V

$$H_\alpha := \mathrm{Fix}(s_\alpha) = \{\lambda \in V : B(\lambda, \alpha) = 0\}.$$

Since W is finite, B is non-degenerate (Theorem 27) and so the map $\alpha \to H_\alpha$ is injective over Φ^+. This implies that there is a bijection between the positive (alternatively, negative) roots Φ^+ and the collection of hyperplanes $\{H_\alpha : \alpha \in \Phi\}$ associated with W. In summary,

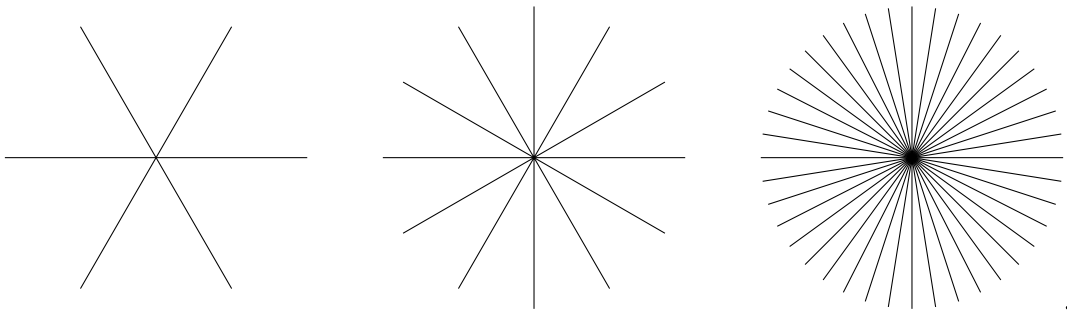
$$\left\{ \begin{array}{c} \text{Hyperplanes of } W \text{ in } V \\ H_\alpha \end{array} \right\} \longleftrightarrow \Phi^+ \longleftrightarrow \left\{ \begin{array}{c} \text{Reflections of } W \text{ acting on } V \\ s_\alpha : V \to V \end{array} \right\}.$$

An important space for what follows in Chapter 4 is what is called the **hyperplane complement** of W

$$V - \bigcup_{\alpha \in \Phi} H_\alpha$$

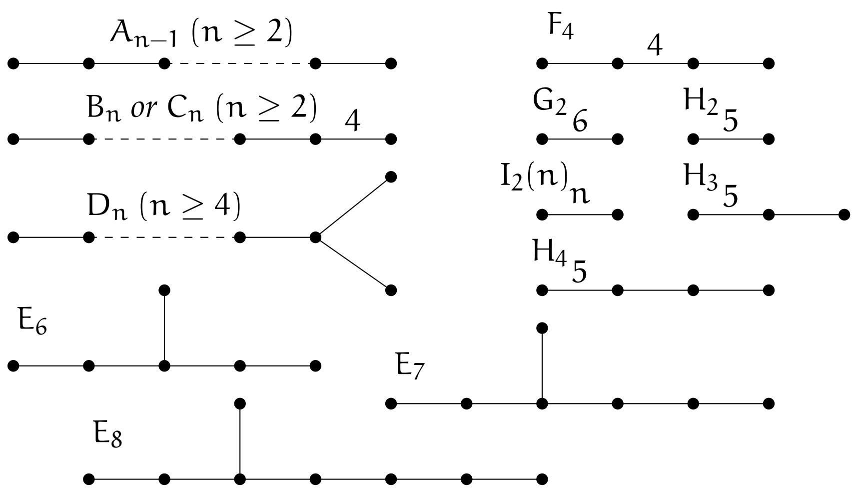
We say that the connected components of the hyperplane complement are **chambers** of the complement.

Example 29. We illustrate the hyperplane complement of $I_2(3)$, $I_2(6)$ and $I_2(20)$ respectively in $\mathbb{R}^2$


.

3.6. **Classification of finite Coxeter groups.** Let Γ be a Coxeter graph. We say W_Γ is **irreducible** if Γ is connected. In this section, we sketch a proof of Coxeter's classification of the finite irreducible Coxeter groups. This classifies the finite Coxeter groups.

Theorem 30 (Coxeter [8], Main Result)**.** *Let Γ be a connected Coxeter graph. W_Γ is finite if and only if Γ is one of the following:*



The index of each type of graph denotes the number of its vertices. Moreover, $I_2(n)$ is a distinct type for $n \geq 7$, but it is conventional to refer to the type $I_2(n)$ for $n \geq 3$ when referring to the dihedral groups.

Proof outline. The moral of the story is that the positivity condition on B can be translated into combinatorial restrictions on the Coxeter graphs specifying finite Coxeter groups. A model example of this is that if Γ specifies an irreducible finite Coxeter group W_Γ, then Γ must be a tree. Otherwise, there would exist a circuit $s_1, s_2, \ldots, s_r$ $(r \geq 3)$ in Γ. Letting $\lambda := \sum_{i=1}^{r} \alpha_{s_i}$, we observe that

$$B(\lambda, \lambda) \leq \sum_{i=1}^{r} B(\alpha_{s_i}, \alpha_{s_i}) + 2 \sum_{i-j \,(\mathrm{mod}\ r) = 1} B(\alpha_{s_i}, \alpha_{s_j}) = r + 2 \sum_{i-j \,(\mathrm{mod}\ r) = 1} B(\alpha_{s_i}, \alpha_{s_j}).$$

since $B(\alpha_{s_i}, \alpha_{s_j}) \leq 0$ if $i - j \pmod r > 1$. Moreover, if $i - j \pmod r = 1$, then $m_{ij} \geq 3$ and so

$$B(\alpha_{s_i}, \alpha_{s_j}) = -\cos\left(\frac{\pi}{m_{ij}}\right) \leq -\frac{1}{2}.$$

Thus, $B(\lambda, \lambda) \leq r - r = 0$, contradicting the fact that B is positive definite. Continuing in this manner, one can successively restrict the pool of potential Coxeter graphs until one ends up with the family of graphs above (see Bourbaki [5] Chapter 5 Section 4.1 for a more detailed treatment).

In the other direction, we note that every graph above is a Coxeter graph since there are no loops and every label is contained in the set $\{4, 5 \ldots, \infty\}$. Thus, to show that every Coxeter graph of the above type specifies a finite Coxeter group, it is sufficient to demonstrate that B is positive definite over the respective reflection representation. $\square$

3.7. **Artin braid groups B_W.** Having developed some of the foundational theory of Coxeter groups, we turn now towards generalising the braid groups introduced in Chapter 2. Just as the braid groups could be obtained from the symmetric group via a group extension, we apply the same philosophy to obtain the Artin braid groups from their associated Coxeter groups.

Given a Coxeter system (W, S), we define the associated **Artin braid group** to be

$$B_W := \langle S \mid R \rangle.$$

where R is the set of non-involution relations of (W, S) as outlined in §3.1. Hence, we obtain a canonical quotient map $B_W \twoheadrightarrow W$ whose kernel we denote PB_W and call the **pure Artin braid group** associated with (W, S). We therefore obtain the following short exact sequence

$$1 \longrightarrow PB_W \longrightarrow B_W \longrightarrow W \longrightarrow 1.$$

Note that $B_{S_n} = \mathcal{B}_n$. We are thus one step closer to articulating the results discussed in §2.4-2.5 in a more general context. What remains now is to extend the configuration space concept to the Artin braid groups.

4. Configuration spaces

In Chapter 3, we introduced some foundational theory of Coxeter groups which allowed us to extend the braid groups to a more general class of groups called Artin braid groups (see §3.7). The goal is to understand whether the results discussed in §2.4-2.5 extend to the Artin braid groups. To do this, we need to develop an analogous object to the configuration spaces $\mathcal{C}_n(\mathbb{C})$ in the braid group context. In this section, we start by observing how $\mathcal{C}_n(\mathbb{C})$ is deeply related to the root system of the symmetric group (§4.1), which will be our motivating example for extending configuration spaces to topological manifolds related to the root systems of arbitrary Coxeter groups (§4.2-4.3). We then formally state the $K(\pi, 1)$ conjecture for Artin braids groups in §4.4.

4.1. Relationship between $\mathcal{C}_n(\mathbb{C})$ and the hyperplanes of S_n.

Recall that the hyperplane complement of the symmetric group is the disconnected space

$$V - \bigcup_{\alpha \in \Phi} H_\alpha$$

where V is the reflection representation of S_n, Φ is the root system of S_n in V and H_α is a hyperplane indexed by a root. In this subsection, we assume the hyperplane complement is always that of the symmetric group and we will prove the following.

Proposition 31. *The configuration space $\mathcal{C}_n(\mathbb{C})$ is homotopy equivalent to*

$$V^{\mathbb{C}} - \bigcup_{\alpha \in \Phi} H^{\mathbb{C}}_\alpha$$

where $V^{\mathbb{C}} := V \otimes \mathbb{C}$ and similarly for $H^{\mathbb{C}}_\alpha$.

Proof. For the sake of visual illustration, we prove everything in the case that $n = 3$, althought the argument is easily modified in general. The symmetric group has two generators s and t that act on $\mathbb{R}^2$ as reflections. Recall from §2.2 that these generators correspond with the adjacent transpositions ($s \mapsto (1,2)$ and $t \mapsto (2,3)$) of the set $\{1,2,3\}$ which act as reflections on $\mathbb{R}^3$ by permuting coordinates and fixing the hyperplanes H_{12} and H_{23} respectively. Since the conjugacy classes of the symmetric group are determined by cycle type, we know that the reflections $\{(1,2),(2,3),(1,3)\}$ make up the complete set of reflections of S_3. Noting that the hyperplane fixed by $(1,3)$ is H_{13}, we consider

$$\bigcup_{i \neq j} H_{ij} = H_{12} \cup H_{23} \cup H_{13}$$

to be the arrangement of hyperplanes in $\mathbb{R}^3$ associated with S_3 (Figure 25).

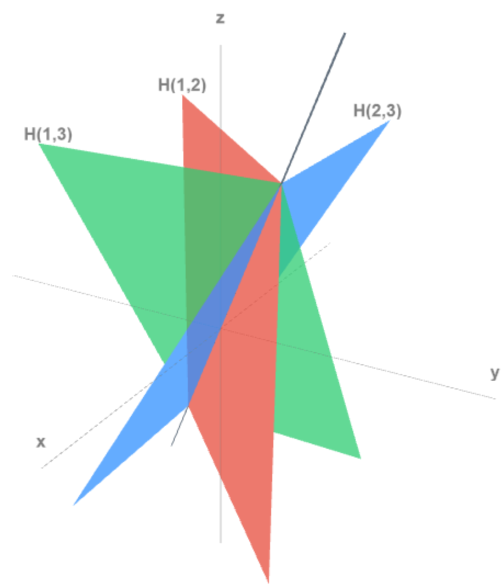


FIGURE 25. Fixed points of the permutation action of S_3 on $\mathbb{R}^3$

There is redundancy in the permutation representation $\mathbb{R}^3$ of S_3. Observe that the action of S_3 along the line $\mathbb{R}(1,1,1)$ is trivial. Consider the decomposition $\mathbb{R}^3 = U \oplus \mathbb{R}(1,1,1)$ where $U = \{x \in \mathbb{R}^n : x_1 + x_2 + x_3 = 0\}$. There is a linear isomorphism $f : V \to U$ given by $\alpha_s \mapsto (1,-1,0)$ and $\alpha_t \mapsto (0,1,-1)$. Moreover, it can be shown that f is equivariant with respect to the action of S_3. As such, f identifies the hyperplanes of S_n (Figure 26) with the hyperplane arrangement in $\mathbb{R}^3$ intersecting U. Consequently,

$$V - \bigcup_{\alpha \in \Phi} H_\alpha \simeq U - \bigcup_{i \neq j} H_{ij}.$$

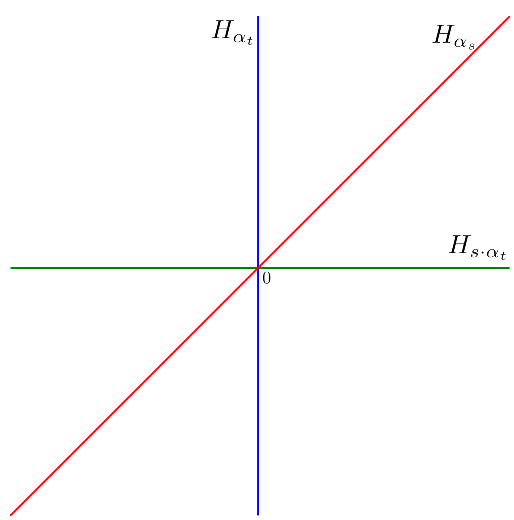


FIGURE 26. Hyperplanes of S_3.

This implies that

$$\mathbb{R}^3 - \bigcup_{i \neq j} H_{ij} \simeq \left(U - \bigcup_{i \neq j} H_{ij} \right) \times \mathbb{R} \simeq \left(V - \bigcup_{\alpha \in \Phi} H_\alpha \right) \times \mathbb{R}.$$

Analogously, by considering the complexified action of S_3 on $V^{\mathbb{C}}$, we obtain the homeomorphism

$$\mathcal{C}_n(\mathbb{C}) \simeq \left(V^{\mathbb{C}} - \bigcup_{\alpha \in \Phi} H_\alpha^{\mathbb{C}} \right) \times \mathbb{C}.$$

Since $\mathbb{C}$ is contractible, we are done. □

4.2. **Configuration spaces Y_W for finite W.** In this section, we extend the configuration space concept to Artin braid groups of finite type. We discus some of the topological properties of these spaces, and describe a way of visualising points in the space.

Proposition 31 relates $\mathcal{C}_n(\mathbb{C})$ to the root system of the symmetric group. This observation provides a rationale for extending the configuration space concept, at least for finite root systems. Namely, given an Artin braid group B_W of finite type, we define its **configuration space** to be

$$Y_W := V^{\mathbb{C}} - \bigcup_{\alpha \in \Phi} H_\alpha^{\mathbb{C}} \tag{4.1}$$

where Φ is the root system of W living in the reflection representation V. Since Φ is finite, the hyperplane arrangement $\{H_\alpha : \alpha \in \Phi^+\}$ is also finite and so for any point x in the hyperplane complement

$$V - \bigcup_{\alpha \in \Phi} H_\alpha$$

there exists an open neighbourhood U, such that if $w \cdot U \cap U$ is nonempty, then $w = 1$. We say that W acts as a **covering space action** because it implies that

$$p' : V - \bigcup_{\alpha \in \Phi} H_\alpha \longrightarrow \left(V - \bigcup_{\alpha \in \Phi} H_\alpha\right)/W$$

is a covering space projection. By the complexified action of W on Y_W, it therefore follows that $p : Y_W \to Y_W/W$ is a covering space projection. Moreover, since Y_W is a manifold, it follows that Y_W/W is also a manifold which we call the **orbit configuration space** [1].

Unlike the hyperplane complement of W, the configuration space Y_W is hard to visualise for $\dim V \geq 2$. Since Coxeter groups are finitely generated, their reflection representations are finite-dimensional. As such, by fixing a basis we may regard $V = \mathbb{R}^n$ and $V^{\mathbb{C}} = \mathbb{C}^n$. We give a method of visualising Y_W in the case that $n = 2$, although it is natural to see how this extends to all $n \geq 1$. We observe that any $\lambda \in \bigcup_\alpha H_\alpha^{\mathbb{C}}$ has the form

$$\lambda = (a + ib) \otimes (x, y) = 1 \otimes (ax, ay) + i \otimes (bx.by) \tag{4.2}$$

[1]for more details on the theory of covering space actions, see §1.3.3 in [15]

where $a, b \in \mathbb{R}$ and $(x, y) \in H_\alpha$. Define the *real part* mapping $\mathfrak{R} : V^{\mathbb{C}} \to V$, $(a + ib) \otimes v = av$, which is well-defined since the map $\mathbb{C} \times V \to V$, $(a + bi, v) \mapsto av$ is $\mathbb{R}$-bilinear. Similarly, we can define the *imaginary part* mapping $\mathfrak{I} : V^{\mathbb{C}} \to V$, $(a + bi) \otimes v = bv$. Then by equation (4.2)

$$\lambda \in \bigcup_{\alpha \in \Phi} H_\alpha^{\mathbb{C}} \iff \exists\, \alpha \in \Phi \text{ such that } \mathfrak{R}(\lambda), \mathfrak{I}(\lambda) \in H_\alpha. \tag{4.3}$$

Therefore, we can determine whether $\lambda \in V^{\mathbb{C}}$ is an element of Y_W by checking the projections $\mathfrak{R}(\lambda)$ and $\mathfrak{I}(\lambda)$. Consider a copy of V, with a pointed arrow $(\longrightarrow)$ with source $\mathfrak{R}(\lambda)$ and target $\mathfrak{I}(\lambda)$, then by (4.3) we obtain a bijection

$$Y_W \leftrightarrow \{\text{Arrows in } \mathbb{R}^2 \text{ with source and target not on the same hyperplane}\}.$$

Figure 27 depicts some of the elements of $Y_{I_2(3)}$.

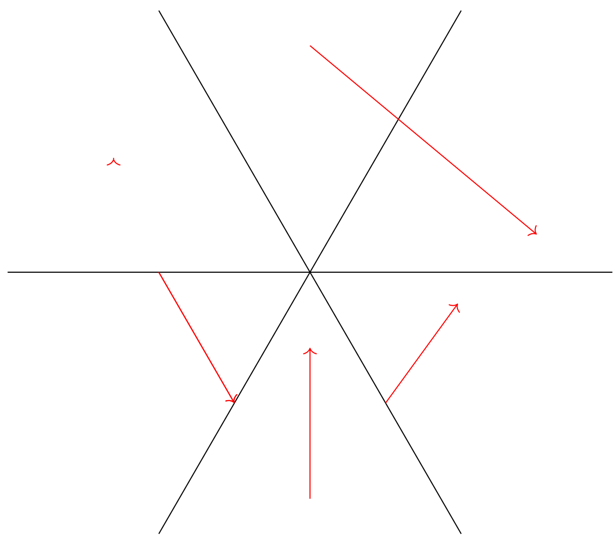
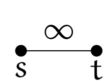


FIGURE 27. Representation of elements in $Y_{I_2(3)}$.

4.3. **Configuration spaces Y_W for infinite W.** In this section we discuss some of the problems that occur for extending the configuration space concept for Artin braid groups of infinite type. To resolve these problems we introduce the Tits cone which realises the hyperplane complement of W as the W-orbit of an open simplicial cone in the daul space V^*. We then define an analogue for Y_W when W is infinite and show that it reduces to the previous definition when W is finite.

Example 32. To motivate this discussion, we naively attempt to work with the hyperplane complement of the infinite dihedral group $I_2(\infty)$. Recall that I_∞ is specified by the Coxeter graph



and so its reflection representation V is 2-dimensional with symmetric bilinear form given by

$$B(\alpha_s, \alpha_s) = B(\alpha_t, \alpha_t) = 1 \quad \text{and} \quad B(\alpha_s, \alpha_t) = -1.$$

Let x and y be real scalars, and observe that

$$s \cdot (x\alpha_s + y\alpha_t) = x\alpha_s + y\alpha_t - 2B(x\alpha_s + y\alpha_t, \alpha_s)\alpha_s = (-x + 2y)\alpha_s + y\alpha_t; \text{ and}$$
$$t \cdot (x\alpha_s + y\alpha_t) = x\alpha_s + y\alpha_t - 2B(x\alpha_s + y\alpha_t, \alpha_t)\alpha_t = x\alpha_s + (2x - y)\alpha_t.$$

As such,

$$x\alpha_s + y\alpha_t \in \mathrm{Fix}(s) \iff x = y \quad \text{and} \quad x\alpha_s + y\alpha_t \in \mathrm{Fix}(t) \iff x = y.$$

But this is disastrous: the positive roots α_s and α_t are mapped to the same hyperplane by the correspondence in §3.5. In other words, the bilinear form B is degenerate, collapsing distinct roots to the same hyperplanes, which was not able to occur in the finite case (see Theorem 27).

The problem is made even more extreme in this case. Given any root α of $I_2(\infty)$, we know that $H_\alpha = w \cdot H_s$ or $H_\alpha = w \cdot H_t$ for some w in $I_2(\infty)$. Since w is generated by s and t, both of which fix H_s and H_t, it follows that the set of hyperplanes associated to the infinite root system of $I_2(\infty)$ is not only finite, it contains only one hyperplane $H_s = H_t$! This is the same hyperplane complement that arises for the finite Coxeter group of type A_2 (see Figure 28). In this sense, the chambers of $Y_{I_2(\infty)}$ fail to capture the algebraic structure of $I_2(\infty)$.

The problem identified in Example 32 is that when W is infinite, B may become degenerate and as a consequence, the correspondence between the positive roots Φ^+ and the hyperplanes $\{H_\alpha : \alpha \in \Phi^+\}$ can collapse. One way to recover this correspondence is to consider the **dual representation** of W given by $\sigma^* : W \to GL(V^*)$ where $\sigma^*(w) = (\phi \circ \sigma)(w^{-1})$ and $\phi : GL(V) \to GL(V^*)$ is precomposition. As before, we use the short hand, $w \cdot f = \sigma^*(w)f$. Let $t = wsw^{-1}$ be a reflection of W, then observe that, for any f in V^* and λ in V

$$(t \cdot f)(\lambda) = (\Phi \circ \sigma(t^{-1}))(f)(\lambda) = f(t \cdot \lambda) = f(\lambda) - 2B(\lambda, w \cdot \alpha_s)f(w \cdot \alpha_s).$$

Then, in the dual space V^* we have

$$f \in \mathrm{Fix}(t) \iff f(w \cdot \alpha_s) = 0.$$

Hence, the associated hyperplane in the dual space of the positive root α_s is given by

$$H_s := \mathrm{Fix}(s) = \{f \in V^* : f(\alpha_s) = 0\}.$$

Moreover, for a positive root $\alpha = w \cdot \alpha_s$, we have

$$H_\alpha := \mathrm{Fix}(ws^{-1}w) = \{f \in V^* : f(\alpha) = 0\} = wH_s.$$

Notice that these hyperplanes no longer depend on B. This gives us the following result without any restriction on W.

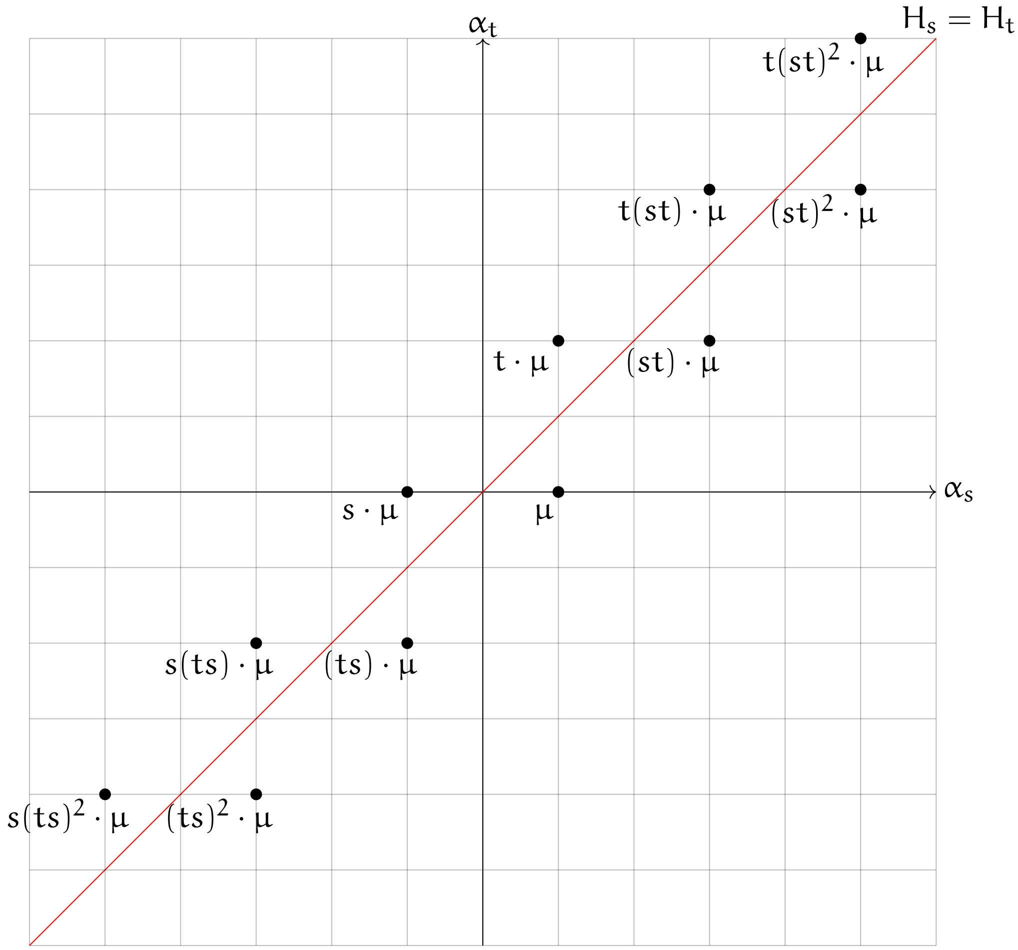


FIGURE 28. Orbit of $\mu = (1, 0)$ in reflection representation of $I_2(\infty)$.

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Proposition 33. *Let α and β be roots in Φ. Then*

$$H_\alpha = H_\beta \implies \alpha = \beta.$$

Proof. Given $s \in S$ and $\alpha \in \Phi$, it suffices to show that

$$H_s = H_\alpha \implies \alpha = \alpha_s.$$

If $|S| = 1$, then $W \simeq \mathbb{Z}/2\mathbb{Z}$ which has one positive root α_s and so the statement holds trivially. Assume $|S| > 1$, then for any generator $t \neq s$ we define the **coroot** $\alpha_t^\vee : V \to \mathbb{R}$ given by $\alpha_t \mapsto 1$ and $\alpha_s \mapsto 0$ for all $s \neq t$. Thus $\alpha_t^\vee \in H_s = H_\alpha$ and so $\alpha - \alpha_s \in \ker \alpha_t^\vee$. We write $\alpha = \sum_{x \in S} c_x \alpha_x$ and observe that

$$0 = \alpha_t^\vee(\alpha - \alpha_s) = \sum_{x \in S} c_x \alpha_t^\vee(\alpha_x) = c_t$$

for all $t \neq s$. Thus $\alpha = c\alpha_s$ for some real number c. Since α and α_s are positive roots, it follows that $c = 1$, and we are done. $\square$

That is, we recover a correspondence between positive roots and hyperplanes in V^*

$$\Phi^+ \longleftrightarrow \{H_\alpha : \alpha \in \Phi^+\}.$$

Example 34. We now construct the hyperplane arrangement of $I_2(\infty)$ in V^*. We pick a basis $\{\alpha_s^\vee, \alpha_t^\vee\}$ of V^* and observe that

$$H_s = \operatorname{span}_{\mathbb{R}}\{\alpha_t^\vee\} \quad \text{and} \quad H_t = \operatorname{span}_{\mathbb{R}}\{\alpha_s^\vee\}.$$

One can verify that

$$(st)^n \cdot \alpha_s = (2n+1)\alpha_s - 2n\alpha_t \quad \text{and} \quad (ts)^{n-1}t \cdot \alpha_s = (2n-1)\alpha_s - 2n\alpha_t;$$
$$(ts)^n \cdot \alpha_t = -2n\alpha_s + (2n+1)\alpha_t \quad \text{and} \quad (st)^{n-1}s \cdot \alpha_t = -2n\alpha_s + (2n-1)\alpha_t.$$

for all $n \in \{1, 2, \ldots\}$. And so, it follows that

$$(st)^n H_s = \operatorname{span}_{\mathbb{R}}\{\alpha_s^\vee + (1 + 1/2n)\alpha_t^\vee\} \quad \text{and} \quad (ts)^{n-1}tH_s = \operatorname{span}_{\mathbb{R}}\{\alpha_s^\vee + (1 - 1/2n)\alpha_t^\vee\};$$
$$(ts)^n H_t = \operatorname{span}_{\mathbb{R}}\{(1 + 1/2n)\alpha_s^\vee + \alpha_t^\vee\} \quad \text{and} \quad (st)^{n-1}sH_t = \operatorname{span}_{\mathbb{R}}\{(1 - 1/2n)\alpha_s^\vee + \alpha_t^\vee\}.$$

We graph some of these hyperplanes in Figure 29.

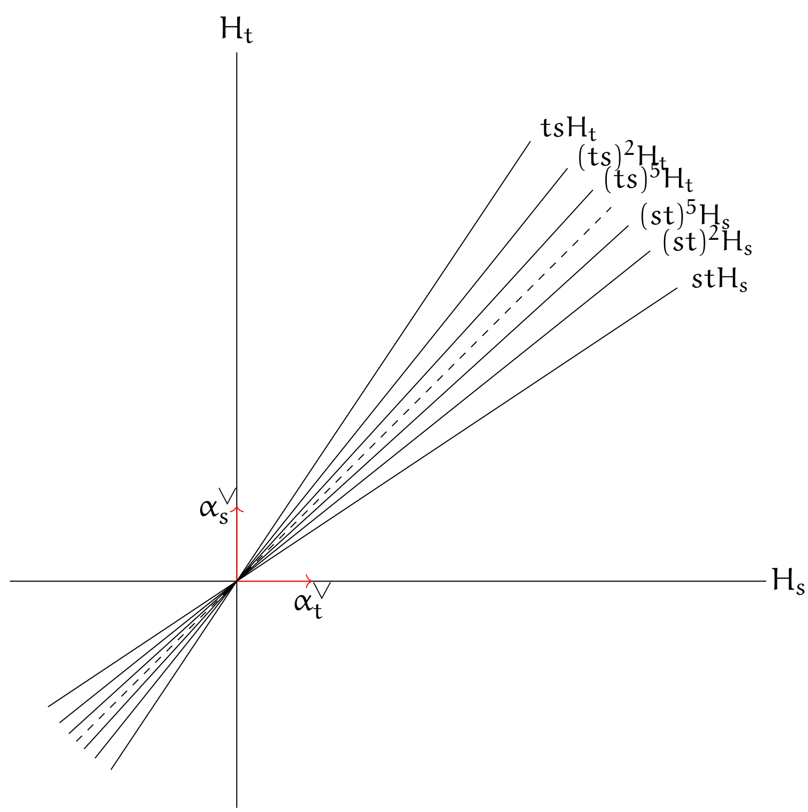


FIGURE 29. A selection of hyperplanes in the reflection arrangement of $I_2(\infty)$ in V^*.

In the limit $n \to \infty$ every hyperplane converges to the limiting hyperplane given by the span of $\alpha_s^\vee + \alpha_t^\vee$. The orbits of points in the hyperplane complement can start to accumulate. In particular, given a point $\lambda \in \operatorname{span}_{\mathbb{R}}\{\alpha_s^\vee + \alpha_t^\vee\}$, for any open neighbourhood U of λ, there exists a hyperplane wH_s which intersects U and the reflection of λ through wH_s is contained in U. In other words, $I_2(\infty)$ fails to act as a covering space action on its hyperplane complement in V^*. This is disastrous since we cannot assure Y_W/W is a manifold when W is infinite.

The problem identified in Example 34 is that when W is infinite, hyperplanes can accumulate towards a limiting hyperplane. Equivalently, we can say that the hyperplane arrangement fails to be locally finite (as in [27]). One solution to this problem is to simply "cut out" the limiting hyperplanes. Let us make this precise: We start by picking a chamber of the hyperplane complement of W called the **fundamental chamber** defined by

$$A_0 := \{f \in V^* : f(\alpha_s) > 0 \text{ for all } s \in S\} = \bigcap_{s \in S} \overbrace{\{f \in V^* : f(\alpha_s) > 0\}}^{:=A_s}$$

which is an open convex cone cut out by the hyperplanes intersecting $\overline{A_0}$. We then consider the orbit of $\overline{A_0}$ by W; that is,

$$\mathcal{T} := \bigcup_{w \in W} w\overline{A_0}$$

which itself is a cone in V^*, called the **Tits cone** [18]. Importantly, $\mathcal{T}$ is the largest region of V^* which is locally finite with respect to the hyperplane arrangement. Moreover, $\mathcal{T}$ is W-stable with chambers in correspondence with the elements of W. Thus, given any Artin braid group B_W, we define its **configuration space** to be

$$Y_W := \mathcal{T} \times \mathcal{T} - \bigcup_{\alpha \in \Phi} H_\alpha^{\mathbb{C}}.$$

In fact, this construction is a genuine generalisation of (4.2) due to the following proposition.

Lemma 35. *Let $s \in S$ and $w \in W$. Then*

$$\ell(sw) > \ell(w) \iff wA_0 \subset A_s$$

and

$$\ell(sw) < \ell(w) \iff wA_0 \subset -A_s.$$

Proof. We prove the first equivalence. The second follows by a similar argument.

($\implies$) We have, by Theorem 24

$$\ell(w^{-1}s) = \ell(sw) > \ell(w) = \ell(w^{-1}) \implies w^{-1} \cdot \alpha_s > 0.$$

So, if $f \in A_0$ then

$$(w \cdot f)(\alpha_s) = f(w^{-1} \cdot \alpha_s) = \sum_s c_s f(\alpha_s)$$

where $c_s \geq 0$ for all $s \in S$. Thus $(w \cdot f)(\alpha_s) > 0$, so $w \cdot f \in A_s$.

$(\Longleftarrow)$ For the sake of contradiction assume

$$w^{-1} \cdot \alpha_s = \sum_{s \in S} c_s \alpha_s < 0.$$

Then, since $\alpha_s^\vee \in A_0$, it follows that

$$(w \cdot \alpha_s^\vee)(\alpha_s) > 0 \implies \alpha_s^\vee(w^{-1} \cdot \alpha_s) > 0 \implies c_s > 0.$$

By this contradicts the negativity of the root $w^{-1} \cdot \alpha_s$. Hence,

$$\ell(sw) = \ell(w^{-1}s) > \ell(w^{-1}) = \ell(w).$$

□

Proposition 36. *Let W be a Coxeter group. Then*

$$\mathcal{T} = V^* \iff W \text{ is finite.}$$

Proof sketch. $(\implies)$ There exists $w \in W$ such that

$$wA_0 = -A_0 = \bigcap_{s \in S} (-A_s) \subset -A_s$$

for all $s \in S$. By Lemma 35, $\ell(sw) < \ell(w)$ for all $s \in S$. Hence

$$\ell(w^{-1}s) = \ell(sw) < \ell(w) = \ell(w^{-1}) \implies w^{-1} \cdot \alpha_s < 0 \quad \text{for all } s \in S.$$

Hence by Proposition 26(ii),

$$|\Phi^+| = \ell(w^{-1}) < \infty \implies |\mathrm{Sym}(\Phi)| < \infty.$$

Since the reflection representation is faithful, the action of W on $|\Phi| < \infty$ is faithful, so $|W| \leq |\mathrm{Sym}(\Phi)| < \infty$.

$(\Longleftarrow)$ This direction requires heavier machinery. For more details see §6.4 in [18]. □

4.4. **The $K(\pi, 1)$ conjecture.** We are now in a position to rephrase the results discussed in §2.4-2.5 in the more general setting of Artin braid groups. That is, we can now formally state the $K(\pi, 1)$ conjecture.

Conjecture ($K(\pi, 1)$ conjecture)**.** *Let B_W be an Artin braid group. Then Y_W/W is $K(B_W, 1)$.*

In §1.6 we discussed, in more detail, the historical context and progress made on solving the conjecture. For Artin braid groups of finite type, this conjecture was solved by Brieskorn [6] and Deligne [10]. In fact, Deligne's result pertained not just to hyperplane complements arising out of Coxeter groups but from a wider class of arrangements known as simplicial arrangements.

Theorem 37 (Deligne [10], Main Theorem)**.** *Let V be a finite dimensional real vector space, $\mathcal{H}$ a finite set of hyperplanes of V, and $Y := V^{\mathbb{C}} - \bigcup_{H \in \mathcal{H}} H^{\mathbb{C}}$. Suppose that the connected components of $V - \bigcup_{H \in \mathcal{H}} H$ are open simplicial cones. Then Y is $K(\pi, 1)$.*

In Chapter 5 we introduce simplicial arrangements and study the properties of their galleries, and in Chapter 6 we provide an outline of Deligne's proof of Theorem 37.

5. Galleries

In Chapter 2, we were able to use the fibre bundle structure of $\mathcal{C}_n(\mathbb{C})$ to show that its higher homotopy groups were trivial. To extend this result to the Artin braid groups of finite type we need a more general approach that relates the topological characteristics of Y_W with its underlying Coxeter structure. In this chapter we develop the theory of galleries of simplicial hyperplane arrangements. We will see that the category of galleries, which capture the algebraic features of hyperplane arrangements, is equivalent as categories to the fundamental groupoid of Y_W, a topological invariant which encodes path-theoretic data of Y_W.

In this Chapter, we assume W is finite.

5.1. **Fundamental groupoid $\Pi(X)$.** The space Y_W is a connected topological manifold. It is therefore path-connected, locally path-connected and locally simply-connected. As such, Y_W has a universal covering space $\widetilde{X}$ built up by homotopy classes of paths in Y_W. To streamline the theory moving forward, we want a topological invariant that packets together the data about the paths in a topological space in such a way that avoids tedious aspects of "base point-picking" common in algebraic topology. This invariant is called the **fundamental groupoid** $\Pi(X)$ of the topological space X.

The fundamental groupoid $\Pi(X)$ is a category whose objects are the points in X and whose morphisms $\mathrm{Hom}(x, y)$ are the homotopy classes of paths with source x and target y. Therefore, the set of endomorphism $\mathrm{End}(x) = \mathrm{Hom}(x, x)$ forms a group structure called the fundamental group $\pi_1(X, x)$ of the based space (X, x). In the context of configuration spaces we are working with connected topological manifolds where the homomorphisms of $\Pi(X)$ resemble the elements of $\pi_1(X)$. In fact, one can regard $\pi_1(X)$ as the category with a single object whose morphisms are the elements of $\pi_1(X)$. Under this perspective, we can obtain the following result.

Proposition 38. *Let X be a path connected space. For each point x in X, the inclusion $\pi_1(X, x) \hookrightarrow \Pi(X)$ is an equivalence of categories.*

Proof. Let $\iota : \pi_1(X, x) \hookrightarrow \Pi(X)$ be the natural inclusion functor. Then ι is faithful and full since the paths γ in $\pi_1(X, x)$ are in correspondence with endomorphisms $\mathrm{End}(x)$ in $\Pi(X)$ with bijection given by $\gamma \mapsto \iota(\gamma)$. Moreover, ι is essentially surjective on objects since path connectedness of X guarantees a path between $\iota(\cdot)$ and y for any $y \in X$, and so $\iota(\cdot)$ is isomorphic to every object in $\Pi(X)$. By a standard result in category theory, we have proven an equivalent condition for equivalence of categories (see, for example, Theorem 1.5.9 in 'Category Theory in Context' [28]) $\square$

5.2. **Pregralleries PGal.** We want to connect the algebraic structure of Coxeter groups W with the topological structure of the fundamental groupoid $\Pi(Y_W)$. Let us first consider what paths look like in Y_W. Recall from §4.2 that a path in Y_W can be regarded as two paths in V, the real and imaginary parts of points in Y_W, that never both lie on the same hyperplane in $\{H_\alpha : \alpha \in \Phi\}$. As such, any path whose real part passes through a hyperplane M has one of the following orientations (see Figure 31).

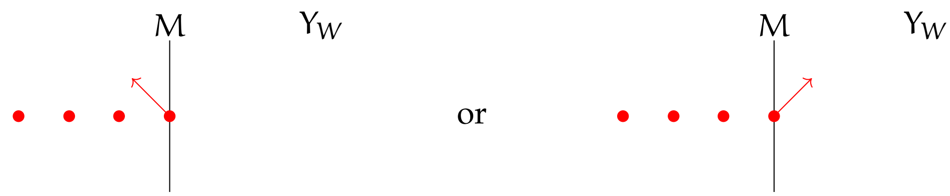


FIGURE 30. Paths in Y_W pass through hyperplanes with an orientation.

In this sense, when the real part crosses a hyperplane such a path in Y_W locally resembles a path around an annulus as in Figure 31.

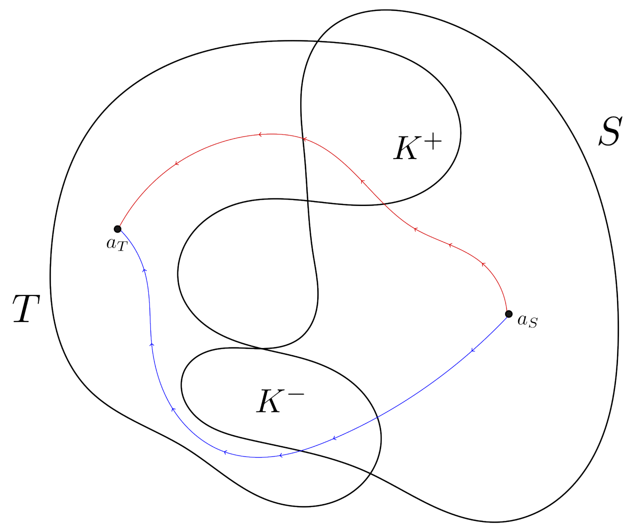


FIGURE 31. Paths passing around

There are essentially four components in this situation: two overlapping open simply connected sets S and T and the two simply connected components of $S \cap T$ which we denote by K^+ and K^-. The goal for this section is to describe $\Pi(Y_W)$ in terms of these four ingredients.

Let $\mathcal{S}$ be an open cover of a topological space $X = \bigcup_{S \in \mathcal{S}} S$ with each S simply connected. For each $S \in \mathcal{S}$, we choose a point $a_S \in S$ and denote the set $A_{\mathcal{S}} := \{a_S : S \in \mathcal{S}\}$. We denote $\Pi := \Pi_1(X, A_{\mathcal{S}})$ to be the full subgroupoid with objects in $A_{\mathcal{S}}$.

Following van der Lek's notation [21], we define A **pregallery** G with respect to $\mathcal{S}$ to be a sequence of elements in $\mathcal{S}$, $(S_0, S_1, \dots, S_k)$ such that $S_{i-1} \cap S_i \neq \varnothing$ together with a choice of a connected component $K_i \subseteq S_{i-1} \cap S_i$ for each $i = \{1, \dots, k\}$. We will simply denote a pregallery by $S_0^{K_1} S_1^{K_2} S_2 \cdots {}^{K_k} S_k$; S_0 is called the **source** and S_k is called the **target**. We can clearly compose two pregalleries with matching source and target:

$$(S_0^{K_1} S_1^{K_2} \cdots {}^{K_k} S_k) \circ (S_k^{K_{k+1}} S_{k+1}^{K_{k+2}} \cdots {}^{K_\ell} S_\ell) := S_0^{K_1} S_1^{K_2} \cdots {}^{K_k} S_k^{K_{k+1}} \cdots {}^{K_\ell} S_\ell.$$

We define the following equivalence class on pregalleries.

Definition 39. Given pregalleries G and G', we write $G \sim_R G'$ if there exists a sequence of pregalleries $G = G_0, \dots, G_n = G'$ such that for each $i \in \{1, \dots, n\}$ the pregallery G_{i-1} differs from G_i only in one of two ways:

1. $S \cdots (P^K U^L Q) \cdots T$ and $S \cdots P^M Q \cdots T$ such that $K \cap L \cap M \neq \varnothing$; or
2. $S \cdots P^P P \cdots T$ and $S \cdots P \cdots T$.

The relation $\sim_R$ is an equivalence relation on the pregalleries of $\mathcal{S}$.

Example 40. Let $X = S^1$, then the following (Figure 32) is a simply-connected covering of X and has the property that $S^K P^L T \nsim_R S^M T$.

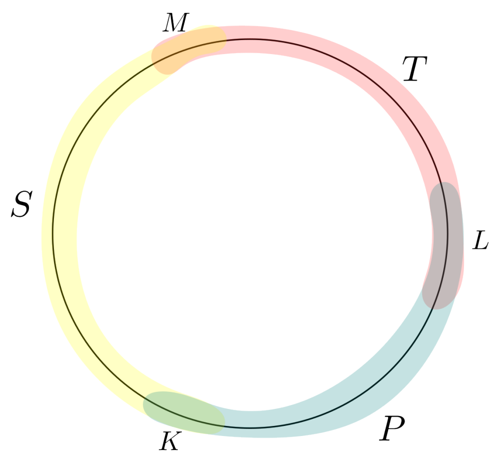


FIGURE 32. $S^K P^L T \nsim_R S^M T$

We let $\mathrm{PGal}(S)$ denote the category whose objects are $S \in \mathcal{S}$ and morphisms $\mathrm{Hom}_{\mathrm{PGal}}(S, T)$ are pregalleries with source S and target T, where an empty pregallery is the identity. Note: PGal is a free category whose morphisms are generated by pregalleries of the form $S^K T$. Our goal is to prove the following theorem.

Theorem 41 (van der Lek [21], Theorem 1.5)**.** *There is an equivalence of categories*

$$\mathrm{PGal}(S) \,/_{\sim_R} \xrightarrow{\simeq} \Pi(X).$$

It is sufficient to construct a functor $\varphi : \mathrm{PGal} \to \Pi$ that is full, essentially faithful on morphisms (that is, $\varphi(G_1) = \varphi(G_2) \implies G_1 \sim_R G_2$) and essentially surjective on objects since $\Pi \hookrightarrow \Pi(X)$ is an equivalence of categories. We define φ in the following way:

On Objects:

$$S \mapsto a_S.$$

On Morphisms:

$$S^K T \mapsto \text{Homotopy class of a path from } a_S \text{ to } a_T \text{ passing through } d \text{ in } K.$$

The first question to ask whether φ depends on my choice of d in K? We answer this in the following proposition.

Proposition 42. *The functor φ is well-defined, full and surjective on objects.*

Proof. To show that the functor φ is well-defined, it is required to be seen that two paths from a_S to a_T that pass via d and d' respectively in K are in the same homotopy class of paths. A path from a_S to a_T via d is the composition of a path γ from a_S to d and a path δ from d to a_T. We can similarly define the path from a_S to a_T via d' as the composition of γ' and δ'. Moreover, since K is path-connected, there exists a path σ from d to d'. This setup is illustrated in Figure 33.

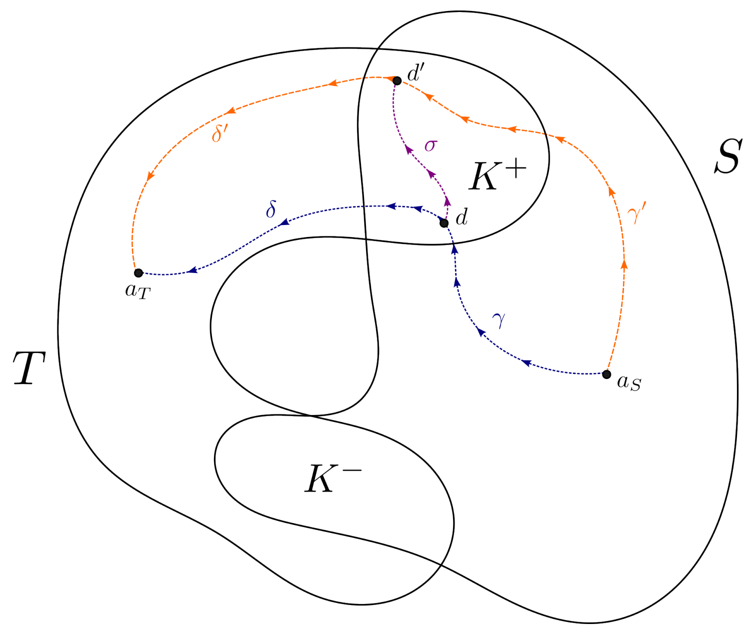


FIGURE 33. PGal $\to \Pi$ is a well-defined functor.

Then, the path $\gamma\sigma$ is a path from a_S to d' contained in S, which is simply connected. Therefore $\gamma\sigma \simeq \gamma'$. Similarly, $\sigma^{-1}\delta \simeq \delta'$. Thus, $\gamma'\delta' \simeq \gamma\sigma\sigma^{-1}\delta \simeq \gamma\sigma$. Hence φ is well-defined, and thus clearly surjective.

Let $\gamma : I \to X$ be a path from a_S to a_T. Since $I = \bigcup_{S \in \mathcal{S}} \gamma^{-1}(S)$ and I is compact, there exists a finite subcover $I = \bigcup_{i=0}^{k} S_i$ and a sequence of points $0 < d_1 < x_1 < \cdots < d_k < 1$ such that $\gamma(x_i) \in S_i$ and $\gamma(d_i) \in K_i \subseteq S_{i-1} \cap S_i$ for each $i \in \{1, \ldots, n\}$, where $S_0 = S$ and $S_k = T$. Then, the pregallery $S^{K_1} S_1^{K_2} \ldots^{K_k} T$, with chosen points $\varphi(d_i) \in K_i$, is mapped by the functor φ to a path in X between a_S and a_T path homotopy equivalent to γ as can be seen by the following figure.

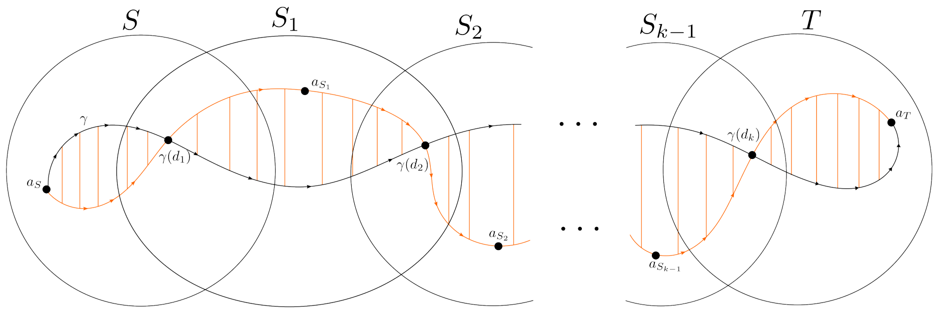


FIGURE 34. PGal $\to \Pi$ is a full functor.

$\square$

We want to characterise the pregalleries in $\mathcal{S}$ which are mapped to the same path homotopy classes under the functor $\varphi : \mathrm{PGal} \to \Pi$.
Let D be a subdivision of $I := [0, 1]$, that is,

$$0 = t_0 < t_1 < \cdots < t_m = 1.$$

Given a path $\gamma : I \to X$ such $\gamma(0) \in S_0$ and $\gamma(1) \in S_m$, we define the set

$$L_\gamma(D) := \{S_0^{K_1} S_1^{K_2} \cdots {}^{K_m} S_m : \gamma([t_i, t_{i+1}]) \subseteq S_i \text{ and } \gamma(t_i) \subseteq K_i, 1 \leq i < m\}.$$

In particular, if $\gamma(0) = a_{S_0}$ and $\gamma(1) = a_{S_m}$, then given $G \in L_\gamma(D)$ it follows that $\gamma \in \varphi(G)$. Moreover, starting with a pregallery G, let γ be a path in $\varphi(G)$ with corresponding partition D (as outlined in Proposition 42), then $G \in L_\gamma(D)$. Furthermore, we define $L_\gamma := \bigcup_D L_\gamma(D)$. We make the following observations about equivalence of pregalleries.

Lemma 43. *The following statements hold:*

1. $G, G' \in L_\gamma(D) \implies G \sim_R G'$*; and*
2. *If* $D' \supseteq D$ *is a refinement of* D*, then for any* $G \in L_\gamma(D)$ *there exists* $G' \in L_\gamma(D')$ *such that* $G \sim_R G'$.

Proof sketch. We prove the second statement. Given the subdivision D expressed as $0 = t_0 < t_1 < \cdots < t_m = 1$, we express D' as $0 = t_0 < \tilde{t}_1 < \tilde{t}_2 < \cdots < \tilde{t}_{M-1} < \tilde{t}_M = 1$, where $m \leq M$. For each $\tilde{t}_j \in D' \setminus D$, there exists $i \in \{0, \ldots, m-1\}$ such that $\tilde{t}_j \in (t_i, t_{i+1})$, and so $\gamma(\tilde{t}_j) \in S_i$. Hence, for any $G = S_0^{K_1} S_1^{K_2} \cdots {}^{K_m} S_m \in L_\gamma(D)$, we can define

$$G' := \left(\prod^{r_0} S_0^{K_1} S_0\right) S_0^{K_1} S_1 \left(\prod^{r_1} S_1^{K_1} S_1\right) S_1^{K_2} S_2 \left(\prod^{r_2} S_2^{K_2} S_2\right) \cdots S_{m-1}^{K_m} S_m \left(\prod^{r_m} S_m^{K_m} S_m\right)$$

where r_i is the number of $\tilde{t}_j \in D' \setminus D$ contained in S_i for all $i \in \{0, \ldots, m\}$. Hence, $G' \in L_\gamma(D')$ and since $S_i^{K_i} S_i \sim_R S_i$, it follows that $G \sim_R G'$. $\square$

Therefore, if $G, G' \in L_\gamma$ then there exists subdivisions D and D' such that $G \in L_\gamma(D)$ and $G' \in L_\gamma(D')$. Hence, there exists $\tilde{G}, \tilde{G}' \in L_\gamma(D \cup D')$ such that

$$G \sim_R \tilde{G} \sim_R \tilde{G}' \sim_R G' \implies G \sim_R G'.$$

Lemma 44. *Let γ and γ' be homotopic as paths and let $G \in L_\gamma$ and $G \in L_{\gamma'}$. Then $G \sim_R G'$.*

Proof. There exists a path homotopy $F : I^2 \to X$ such that $F(x, 0) = \gamma(x)$ and $F(x, 1) = \gamma'(x)$. Since I^2 is compact, we can pass to a finite cover $\{S_i\}_{i=1}^n$ which we can assume contains the open sets in the pregalleries G and G'. Moreover, we can subdivide I^2 into a finite rectangular grid with vertical and horizontal subdivisions given by

$$0 = r_0 < r_1 < \cdots < r_n = 1 \quad \text{and} \quad 0 = c_0 < c_1 < \cdots < c_m = 1$$

respectively, such that $F([c_{i-1}, c_i] \times [r_{j-1}, r_j])$ is contained in some S_k, for all $1 \leq i \leq m$ and $1 \leq j \leq n$. From this grid, we can define a finite sequence of paths $\gamma_j(x) := F(x, r_j)$, where $0 \leq j \leq n$. Suppose $n = 1$, then the regions $F(I \times [c_{i-1}, c_i])$ and $F(I \times [c_i, c_{i+1}])$, where $1 \leq i < m$, are contained in S_k and S_ℓ respectively. Moreover, there is a connected component $K \subseteq S \cap T$ such that $F(I, c_i) \subset K$. Therefore, we can generate a pregallery $\widetilde{G}$ in $L_\gamma \cap L_{\gamma'}$ from the association of $S_k^K S_\ell$ with the region $I \times [c_{i-1}, c_{i+1}]$. By Lemma 43, it follows that $G \sim_R \widetilde{G} \sim_R G'$.

Assume $n \geq 2$. Fix $j \in \{0, \ldots, n-2\}$, then the cells $[r_j, r_{j+1}] \times [c_{i-1}, c_{i+1}]$ and $[r_{j+1}, r_{j+2}] \times [c_{i-1}, c_{i+1}]$ can be associated, in the same way as above, with pregalleries $G_j \in L_{\gamma_j} \cap L_{\gamma_{j+1}}$ and $G_{j+1} \in L_{\gamma_{j+1}} \cap L_{\gamma_{j+2}}$ respectively. Since both $G_j, G_{j+1} \in L_{\gamma_{j+1}}$ it follows by Lemma 43 that $G_j \sim_R G_{j+1}$. Hence

$$G \sim_R G_1 \sim_R \cdots \sim_R G' \implies G \sim_R G'.$$

$\square$

Proof of Theorem 41. ($\Longleftarrow$) It is sufficient to assume that G and G' differ only in the one of two ways listed in Definition 39. If they differ in the second way, it follows that $\varphi(G) = \varphi(G')$ since every $S \in \mathcal{S}$ is simply connected. If they differ in the first way, then there is a pregallery $P^K U^L Q$ in G and $P^M Q$ in G' such that $K \cap L \cap M \neq \varnothing$. Let $d \in K \cap L \cap M$, and let $\varphi_1, \varphi_2, \varphi_3 : I \to X$ be paths in X terminating at d and starting at the points a_P, a_Q and a_U respectively (see Figure 35). Then $\varphi_1\varphi_2^{-1} \in \varphi(G')$ and $\varphi_1\varphi_2^{-1} \simeq (\varphi_1\varphi_3^{-1})(\varphi_3\varphi_2^{-1}) \in \varphi(G)$. Hence $\varphi(G) = \varphi(G')$.

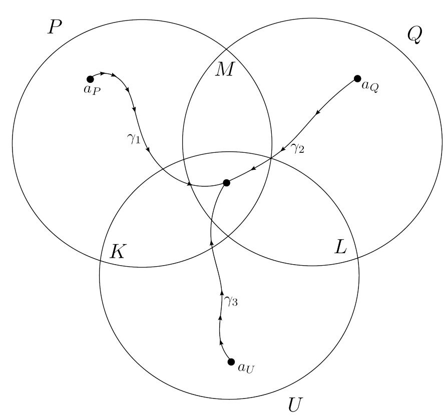


FIGURE 35. $G \sim_R G' \implies \varphi(G) = \varphi(G')$.

($\implies$) The functor φ maps G and G' to the homotopy classes containing paths γ and γ'. Since $\varphi(G) = \varphi(G')$, these paths are path homotopic and $G \in L_\gamma$ and $G' \in L_{\gamma'}$. Therefore, by Lemma 44 $G \sim_R G'$. ☐

5.3. **Chambers and galleries.** In the previous section we described $\Pi(Y_W)$ in terms of pregalleries of a covering of open simply connected sets of Y_W. Pregalleries encode the orientation of paths in Y_W as they move around the complexified hyperplanes $H_\alpha^{\mathbb{C}}$. As such, PGal is already more suitable for studying the underlying Coxeter structure of Y_W than $\Pi(Y_W)$. However, pregalleries do not tell us enough about the rigid geometric features of the hyperplane complements of W. For example, pregalleries do not tell us the number of chambers traversed by the real part of paths in Y_W. Nor can they tell us about the minimal number of chambers between the real parts of two points in Y_W.

For the remainder of this chapter our goal is to construct a category built out of the chambers of hyperplane arrangements. The hope is to relate this category to $\Pi(Y_W)$ via $\mathrm{PGal}(\mathcal{S})$.

Let us start with the anatomy of a hyperplane arrangement $(V, \mathcal{H})$. We define the following equivalence relation on V

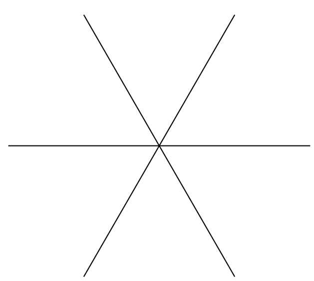
$$x \sim y \iff \forall H \in \mathcal{H},\ \lambda, \mu \in H \text{ or } \lambda, \mu \text{ are strictly on the same side of } H.$$

We call the equivalence classes the **facets** of $(V, \mathcal{H})$. If F is a facet, then we define its **support** to be

$$\mathrm{supp}(F) := \bigcap_{\substack{H \in \mathcal{H} \\ F \subseteq H}} H.$$

As such, we can define the **dimension** of F as $\dim F := \dim \mathrm{supp}(F)$. We call the $\dim V$ dimensional facets the **(open) chambers** and the codimension 1 facets the **faces** of $(V, \mathcal{H})$. The support of a face of a chamber is called a **wall**.

Example 45. The following hyperplane arrangement in $\mathbb{R}^2$ has 6 chambers, 6 faces and 1 dimension 0 facet (the origin).



Remark 46. The origin need not always be its own facet, in the hyperplane arrangement associated with the action of S_3 on $\mathbb{R}^3$, the set $\mathrm{span}_{\mathbb{R}}\{(1, 1, 1)\}$ is a facet.

Lemma 47. *Facets are open convex cones.*

Proof. The statement follows immediately from the fact that the open half-spaces of H are convex cones and the intersection of two convex cones is a convex cone. ☐

Lemma 48. $x \sim y \iff$ *there exists a facet* F *such that the line segment joining* x *and* y *is in* F.

Proof. Define $f : I \to V$, $t \mapsto (1-t)x + ty$, then f parameterises the line segment $f(I)$ joining x to y. If $x \sim y$, then there is a facet F containing x and y. By Lemma 47 F is a convex cone and so $f(t) \in F$ for all $t \in I$. That is, $f(I) \subseteq F$. Moreover, if the line segment joining x to y is in a facet F, then in particular $x = f(0) \sim f(1) = y$. $\square$

Lemma 49 (Bourbaki [5], Proposition 5.1.2.3.(ii))**.** *Let* F *be a facet. The closure* $\overline{F}$ *of* F *is a union of facets of dimension strictly smaller than* F.

We want to construct a morphism between the chambers in a hyperplane arrangement like we did with pregalleries for coverings of open simply connected sets. Let C be a chamber in $(V, \mathcal{H})$. A facet of C is a facet $F \subseteq \overline{C}$. Two chambers are **adjacent** if they share a face. We define a **gallery** G from A (**source**) to B (**target**) of length n to be a sequence of chambers $A = C_0, C_1, \ldots, C_n = B$, which we denote by $G = C_0C_1 \cdots C_n$, such that C_{i-1} and C_i are adjacent for all $1 \le i \le n$. For convenience of notation, we may denote a gallery with source A and target B by $A \xrightarrow{G} B$.

We can compose galleries whose target and source are adjacent by adjoining. Moreover, given a gallery $G = C_0C_1 \cdots C_n$, we have the following special galleries:

(1) (**Opposite gallery**) $G^* := C_n \cdots C_1C_0$; and
(2) (**Antipodic gallery**) $-G := (-C_0)(-C_1) \cdots (-G_n)$.

Let $d(A, B)$ denote the smallest length of all galleries from A to B. Given a gallery from A to B, we say that it is **minimal** if it has length $d(A, B)$. The first question to answer is how can we compute $d(A, B)$? Let us define

$$\mathcal{H}(A, B) := \{\text{all } H \in \mathcal{H} \text{ such that } A \text{ and } B \text{ not strictly on the same side}\}.$$

Then we must have $d(A, B) \ge |\mathcal{H}(A, B)|$ since these hyperplanes must be traversed at least once.

Lemma 50. *Let* A, B, C *be chambers. Then*

$$\mathcal{H}(A, C) = \mathcal{H}(A, B)\Delta\mathcal{H}(B, C) = (\mathcal{H}(A, B)\backslash\mathcal{H}(B, C)) \cup (\mathcal{H}(B, C)\backslash\mathcal{H}(A, B)).$$

Proof. Let $H \in \mathcal{H}(A, C)$. Then one of two configurations occur (Figure 36).

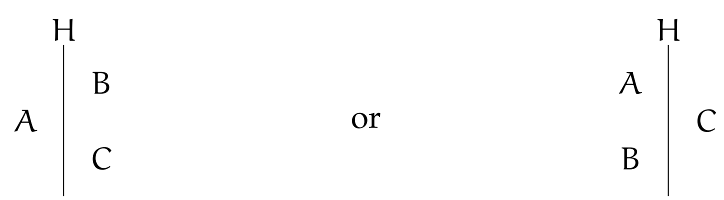


FIGURE 36. A, B, C relative to $H \in \mathcal{H}(A, C)$.

That is, $H \in \mathcal{H}(A, B)\backslash\mathcal{H}(B, C)$ or $H \in \mathcal{H}(B, C)\backslash\mathcal{H}(A, B)$. Moreover, if $H \in \mathcal{H}(A, B)\Delta\mathcal{H}(B, C)$, then one of two configurations occur (Figure 37).

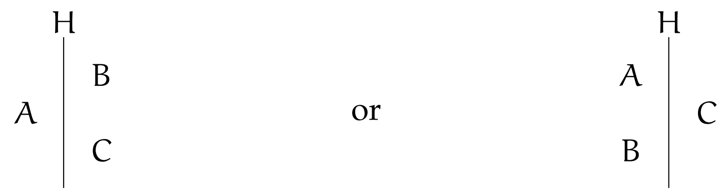


FIGURE 37. A, B, C relative to $H \in \mathcal{H}(A, B)\Delta\mathcal{H}(B, C)$.

In either case, $H \in \mathcal{H}(A, C)$. $\square$

The following proposition tells us that a gallery from A to B is minimal if and only if it crosses all separating hyperplanes of A and B exactly once, and no other hyperplanes.

Proposition 51. *Given chambers A and B*

$$d(A, B) = |\mathcal{H}(A, B)|.$$

Proof. If $|\mathcal{H}(A, B)| = 0$, then $A = B$ and so $d(A, B) = |\mathcal{H}(A, B)|$. Assume for all galleries from A to B such that $0 \leq |\mathcal{H}(A, B)| < n$, it followed that $d(A, B) = |\mathcal{H}(A, B)|$. Now consider chambers A and B such that $|\mathcal{H}(A, B)| = n$. Then we can pick $H \in \mathcal{H}(A, B)$ such that H contains a face shared by A and an adjacent chamber C. Note that $\mathcal{H}(A, C) = \{H\}$, and so by Lemma 50

$$\begin{aligned} \mathcal{H}(A, B) &= (\mathcal{H}(A, C) \setminus \overbrace{\mathcal{H}(C, B)}^{\not\ni H}) \cup (\overbrace{\mathcal{H}(C, B)}^{\not\ni H} \setminus \mathcal{H}(A, C) \\ &= \{H\} \cup \mathcal{H}(C, B). \end{aligned}$$

Therefore, by the inductive hypothesis, $|\mathcal{H}(A, B)| = 1 + d(C, B)$. And so there is a gallery from A to B of length $|\mathcal{H}(A, B)|$ and thus $d(A, B) \leq |\mathcal{H}(A, B)|$. $\square$

We now prove an important structural result that relates the galleries of chambers in a hyperplane arrangement to galleries of chambers in a quotient space. This will be essential in Deligne's argument and is really a consequence of the correspondence theorem for vector spaces.

Theorem 52 (Deligne [10], Prop 1.5)**.** *Let P be an intersection of hyperplanes in $\mathcal{H}$. Denote $\mathrm{pr}_P : V \to V/P$ and define*

$$\begin{aligned} \mathcal{H}_P &:= \{N \subseteq V/P : N \textit{ is a hyperplane in } V/P \textit{ and } \mathrm{pr}_P^{-1}(N) \in \mathcal{H}\} \\ &= \{\mathrm{pr}_P(H) \in V/P : H \in \mathcal{H} \textit{ and } H \supseteq P\}. \end{aligned}$$

Moreover, let $\mathcal{F}$ and $\mathcal{F}_P$ denote the facets of $(V, \mathcal{H})$ and $(V/P, \mathcal{H}_P)$ respectively. We define $\pi'_P : \mathcal{F} \to \mathcal{F}_P$ to be the unique map satisfying $\mathrm{pr}_P(F) \subseteq \pi'_P(F)$. Then the following hold:

1. *Suppose F is a facet with support P. We define the subset of facets*

$$\mathcal{F}(F) := \{E \in \mathcal{F} : \overline{E} \supseteq F\}.$$

Then, π'_P restricted to $\mathcal{F}(F)$ is a bijection.

2. *The bijection in (1.) preserves codimension of facets and incidence relations. Hence, adjacent chambers in $\mathcal{F}(F)$ are adjacent in $\mathcal{F}_P$, and so minimal galleries in $\mathcal{F}(F)$ are sent to minimal galleries in $\mathcal{F}_P$ and vice versa.*

Proof sketch. Note that the equality of sets in the definition of $\mathcal{H}_P$ is a consequence of the correspondence theorem for modules.

We start by proving the existence of π'_P. Note, if such a map exists, then it is the unique map $\mathcal{F} \to \mathcal{F}_P$ satisfying the property $\mathrm{pr}_P(F) \subseteq \pi'_P(F)$ since $\mathcal{F}_P$ partition V/P, and so there cannot be two facets of V/P containing $\mathrm{pr}_P(F)$.

Let $x, y \in \mathrm{pr}_P(F)$. Then there exists $\widetilde{x}, \widetilde{y} \in F$ such that $x = \mathrm{pr}_P(\widetilde{x})$ and $y = \mathrm{pr}_P(\widetilde{y})$. Suppose for all $N \in \mathcal{H}_P$, $x \notin N$. Then $\widetilde{x} \notin \mathrm{pr}_P^{-1}(N)$ for all $N \in \mathcal{H}_P$. Therefore, the facet F does not intersect any $\mathrm{pr}_P^{-1}(N)$ for all $N \in \mathcal{H}_P$. So pr_P sends the line segment I joining $\widetilde{x}$ to $\widetilde{y}$ to the line segment $\mathrm{pr}_P(I)$ joining x and y such that $\mathrm{pr}_P(I) \cap N = \varnothing$ for all $N \in \mathcal{H}_P$. Therefore, by Lemma 48, x and y are in the same chamber. Now assume there exists $N \in \mathcal{H}_P$ such that $x \in N$. Then $\widetilde{x} \in \mathrm{pr}_P^{-1}(N) \in \mathcal{H}$, and so $\widetilde{y} \in \mathrm{pr}_P^{-1}(N)$. Since $\widetilde{x} \sim \widetilde{y}$, $y \in N$ and on the same side as x for any other hyperplane in $\mathcal{H}_P$.

(1. Surjectivity). Let G be a facet in $\mathcal{F}_P$. Then by Lemma 47, G is a cone and so $0 \in \overline{G}$. It is therefore sufficient to show that there exists $\varepsilon > 0$ such that

$$B(\varepsilon, 0) \subseteq \mathrm{pr}_P\left(\bigcup_{E \in \mathcal{F}(F)} E\right)$$

where $B(\varepsilon, 0)$ is the open ε-ball centred at 0. Fix $x \in F$. By Lemma 49, if $x \in \overline{E}$, then $\overline{E} \supseteq F$, and so for any $E \notin \mathcal{F}(F)$, there exists $\varepsilon_E > 0$ such that $B(\varepsilon_E, x) \cap E = \varnothing$. Since $\mathcal{H}$ is finite, there are finitely many facets of $(V, \mathcal{H})$ and so there exists $\varepsilon := \min\{\varepsilon_E : E \in \mathcal{F} - \mathcal{F}(F)\}$, satisfying $B(\varepsilon, x) \subseteq \bigcup_{E \in \mathcal{F}(F)} E$. Hence

$$\mathrm{pr}_P(B(\varepsilon, x)) \subseteq \mathrm{pr}_P\left(\bigcup_{E \in \mathcal{F}(F)} E\right).$$

Since $F \subseteq P$, $0 \in \mathrm{pr}_P(B(\varepsilon, x))$. Moreover, $\mathrm{pr}_P(B(\varepsilon, x))$ is an open set in V/P and so we have $\delta > 0$ such that $\delta < \varepsilon$ and

$$B(\delta, 0) \subset \mathrm{pr}_P(B(\varepsilon, x)) \subseteq \mathrm{pr}_P\left(\bigcup_{E \in \mathcal{F}(F)} E\right).$$

(1. Injectivity). Suppose there exists $E_1, E_2 \in \mathcal{F}(F)$ such that $\pi'_P(E_1) = \pi'_P(E_2)$. First, for any $z \in F$, there exists $\varepsilon > 0$ such that $B(\varepsilon, z) \cap H \neq \varnothing$ if and only if $z \in H$. Moreover, $B(\varepsilon, z) \cap E_1 \neq \varnothing$ and similarly for E_2. Hence, pick $x \in B(\varepsilon, z) \cap E_1$ and $y \in B(\varepsilon, z) \cap E_2$. Thus, x and y are on the same side of all H such that $H \not\supseteq P \supseteq F$.

Since $\mathrm{pr}_P(x) \sim \mathrm{pr}_P(y)$, two possibilities remain. Case 1: $\mathrm{pr}_P(x) \in N$, for some $N \in \mathcal{H}_P$. Then, $x, y \in \mathrm{pr}_P^{-1}(N) \in \mathcal{H}$. Case 2: $\mathrm{pr}_P(x)$ is not in any hyperplane in $\mathcal{H}_P$. Then by correspondence with hyperplanes in V containing P, x and y are on the same side of all hyperplanes in $\mathcal{H}$ such that $H \supseteq P$. That is, $x \sim y$. $\square$

Let P be the support of a facet F and let C be a chamber such that $F \subseteq \overline{C}$. Moreover, we define $\pi_F := \left(\pi'_P\big|_{\mathcal{F}(F)}\right)^{-1}$. Then, we call the following chamber

$$C.\Delta(P) := \pi_F(-\pi'_P(C))$$

the **local opposite chamber** of C.

Corollary 53 (Codimenion 1)**.** *Let H be a supporting hyperplane of a face F such that $F \subset \overline{C}$. Then $C.\Delta(H)$ is the adjacent chamber of C sharing F.*

Proof. Note that $\mathcal{F}(F) = \{C, D\}$ where D is the unique chamber adjacent to C sharing the face F. Moreover, $\dim V/H = 1$ and $\mathcal{H}_H = \{\mathrm{pr}_H(H)\} = \{0\}$. Hence there are two chambers in $V/H - \{0\}$, and so $-\pi'_H(C) = \pi_H(D)$. Thus $C.\Delta(H) = \pi_F(\pi_H(D)) = D$. $\square$

Corollary 54 (Codimenion 2)**.** *Let P be the support of a facet F and suppose C is a chamber such that $F \subset \overline{C}$. If $\operatorname{codim} F = 2$, then there are exactly 2 minimal galleries from C to $C.\Delta(P)$.*

Proof. Note that $\dim V/P = 2$, and so $\pi'_P(C)$ and $-\pi'_P(C)$ are opposite chambers in a 2-dimensional arrangement of hyperplanes (see Figure 38).

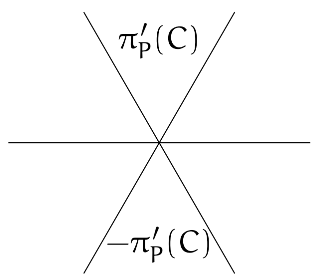


FIGURE 38. Hyperplane arrangement in V/P where $\operatorname{codim} P = 2$.

As such, there are exactly two minimal galleries (i.e., clockwise, anticlockwise) between $\pi'_P(C)$ and $-\pi'_P(C)$ in V/P. By Theorem 52, it therefore follows that there are exactly two minimal galleries between $\pi_F(\pi'_P(C)) = C$ and $\pi_F(-\pi'_P(C)) = C.\Delta(P)$ in V. $\square$

5.4. **Simplicial arrangements.** The hyperplanes associated to a Coxeter group form a special arrangement of hyperplanes called a simplicial arrangement. In this section, we define these arrangements and consider some of their properties.

Let F be an open convex cone in V. We say that F is **simplicial** if there exists a linearly independent set of vectors $v_1, \ldots, v_k$ such that

$$F = \left\{ \sum_{i=1}^{k} a_i v_i : a_i > 0 \text{ for all } i \right\}.$$

If every chamber in a hyperplane arrangement $(V, \mathcal{H})$ is simplicial, we call $(V, \mathcal{H})$ a **simplicial arrangement**.

Example 55. The hyperplane arrangement associated to a Coxeter group is simplicial. This due to the Tits cone construction of the hyperplane complement shown in §4.3,

$$\mathcal{T} = \bigcup_{w \in W} w\overline{A_0}$$

where A_0 is a simplicial chamber of the hyperplane complement.

A defining property of simplicial arrangements is the following.

Proposition 56 (Deligne [10], Remark 1.9.2)**.** *Let $(V, \mathcal{H})$ be a simplicial arrangement. If C be a chamber with two distinct walls M and N, then there exists a unique facet F in C such that $M \cap N$ is the support of F.*

As a consequence, if P is the support of the intersection of two walls M and N of a chamber C, then the local opposite chamber $C.\Delta(P)$ of C is well-defined. This will turn out to be an essential fact for Deligne's proof in Chapter 6.

Example 57. We provide an example of arrangement which is not simplicial. Consider the arrangement of the hyperplanes defined by the equations

$$z = x + y, \quad z = x - y, \quad z = -x + y, \quad z = -x - y$$

in $\mathbb{R}^3$ (see Figure 39).

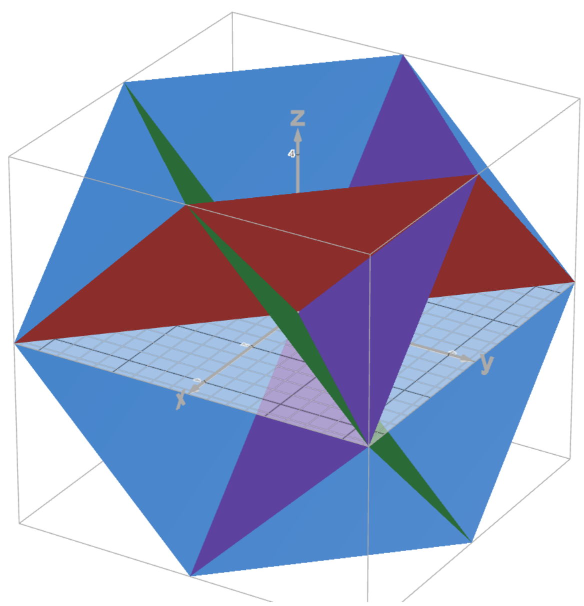


FIGURE 39. A non-simplicial hyperplane arrangement in $\mathbb{R}^3$.

Let C be the chamber containing the point $(0, 0, 1)$ and H denote the hyperplane defined by $z = x + y$. Then H and $-H$ are walls of C and their intersection is the line L defined by $y = -x$ and $z = 0$. There are only two facets whose support is L; namely,

$$F_1 := \{(x, y, z) : y = -x,\ x > 0 \text{ and } z = 0\} \quad \text{and} \quad F_2 := \{(x, y, z) : y = -x,\ x < 0 \text{ and } z = 0\}.$$

But neither F_1 nor F_2 are facets of the chamber C. This contradicts the condition in Proposition 56, and so this hyperplane arrangement is *not* simplicial.

Having a well-defined local opposite chamber $C.\Delta(P)$ will be advantageous since minimal paths between C and $C.\Delta(P)$ are highly controlled.

Lemma 58 (Deligne [10], Lemma 1.7.(iii))**.** *Let $(V, \mathcal{H})$ be a simplicial arrangement. Given distinct walls M and N of a chamber C, there are exactly two minimal galleries from C to $C.\Delta(M \cap N)$. One begins with $(C, C.\Delta(M))$ and the other begins with $(C, C.\Delta(N))$ (see Figure 40).*

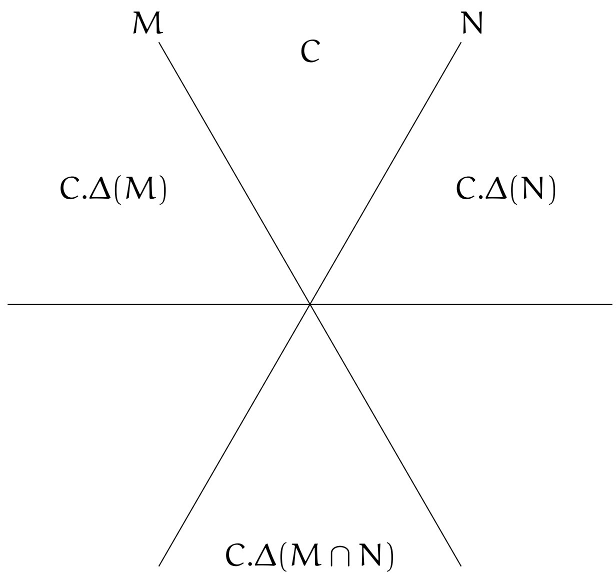


FIGURE 40. Two minimal galleries from C to $C.\Delta(M \cap N)$.

5.5. **The category of galleries** Gal**.** Having established some foundational theory for galleries of simplicial arrangements, we now construct the category of galleries and show that it is equivalent as categories to the fundamental groupoid $\Pi(Y_W)$ when $(V, \mathcal{H})$ is the hyperplane arrangement of W. In doing so, we realise the data packaged by $\Pi(Y_W)$ in terms of the rigid geometric features of $(V, \mathcal{H})$. For the remainder of this chapter, we assume $(V, \mathcal{H})$ is simplicial.

Let Gal_0 be the category whose objects are chambers in $(V, \mathcal{H})$ and morphisms are the galleries between them. Given two galleries with same source G and G', we say they are equivalent $G \sim G'$ if there is a sequence of galleries $G = G_0, \dots, G_n = G'$ for $n \geq 0$ such that G_{j+1} is related to G_j in the following way:

(i) we have the decompositions $G_j = E_1 F E_2$ and $G_{j+1} = E_1 F' E_2$; such that
(ii) there is a chamber C with walls M and N such that F and F' are the two minimal galleries from C to $C.\Delta(M \cap N)$.

We call the progression from G_j to G_{j+1} an **elementary move**. The relation $\sim$ is in fact an equivalence relation, and allows us to the define the quotient category called the **category**

of positive galleries

$$\mathrm{Gal}_+ := \mathrm{Gal}_0 /_{\sim}.$$

Let h_1 and h_2 be equivalence classes of galleries (that is, morphisms in Gal_+) such that the target of h_1 is the source of h_2. We define h_1h_2 to be the equivalence class of galleries from the source of h_1 to the target of h_2 passing through the target of h_1 (or source of h_2). If there exists an equivalence class of galleries g such that $g = h_1h_2$, then we say g **begins** with h_1.

The following proposition tells us that equivalence of galleries behaves like homotopy equivalence of paths.

Proposition 59 (Deligne [10], Proposition 1.19)**.** *The following hold:*

1. *Suppose we have*

$$\begin{tikzcd} A \arrow[r, "F"] & B \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] & C \end{tikzcd}$$

where $FG_1 \sim FG_2$*. Then* $G_1 \sim G_2$*.*

2. *Suppose we have*

$$\begin{tikzcd} A \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] & B \arrow[r, "G"] & C \end{tikzcd}$$

where $F_1G \sim F_2G$*. Then* $F_1 \sim F_2$*.*

3. *Let* g *be an equivalence class of galleries with source* A*. Then there exists a unique chamber* Δ *such that for any chamber* B*,* g *begins with* $u(A, B)$ *if and only if* $\mathcal{H}(A, \Delta) = \mathcal{H}(A, B) \sqcup \mathcal{H}(B, \Delta)$*.*

This is already looking to be a promising model of the fundamental groupoid. What is lacking is that the morphisms are not invertible. To fix this, we define the **category of galleries** Gal to be the groupoid completion of Gal_+. For each g in $\mathrm{Hom}_{\mathrm{Gal}}(A, B)$, we denote its formal inverse g^{-1} in $\mathrm{Hom}_{\mathrm{Gal}}(B, A)$. Note, g^{-1} is not to be mistaken for $-g$ or g^*, all three of these are distinct morphisms in Gal.

Theorem 60 (van der Lek [21], Theorem 4.10)**.** *There is an equivalence of categories*

$$\mathrm{Gal} \xrightarrow{\simeq} \mathrm{PGal}(S) /_{\sim_R} \xrightarrow{\simeq} \Pi(X).$$

A remarkably important algebraic feature of Gal is its *Garside structure*. We will not discuss the importance of this here but a discussion can be found in [9, 4, 12]. However, we do consider some related properties of Gal. Let $g.\Delta := u(B, -B)$ and $\Delta.g := u(-A, A)$ where A is the source of g and B is the target of g.

Lemma 61 (Deligne [10], Lemma 1.26.(i))**.** *In* Gal_+*, for each morphism* g*, we have*

$$g.\Delta = \Delta.(-g).$$

Proof. We express g in the following form

$$g = u(C_0, C_1)u(C_1, C_2) \cdots u(C_{n-1}, C_n).$$

We proceed via induction on n. If $n = 1$, then $g.\Delta = u(C_0, C_1)u(C_1, -C_1)$. Since $u(C_1, -C_1)$ begins with $u(C_1, -C_0)$, it follows that $g.\Delta = u(C_0, C_1)u(C_1, -C_0)u(-C_0, -C_1)$. Moreover, since $u(C_0, -C_0)$ begins with $u(C_0, C_1)$, we thus obtain $g.\Delta = u(C_0, -C_0)u(-C_0, C_1) = \Delta.(-g)$. Assume the lemma holds for all equivalence classes of galleries of length $< n$, then write $g = u(C_0, C_1)h$ and observe that

$$g.\Delta = u(C_0, C_1)(h.\Delta) = u(C_0, C_1)\Delta.(-h) = \Delta.u(-C_0, -C_1)(-h) = \Delta.(-g).$$

$\square$

Theorem 62 (Deligne [10], Proposition 1.27.(ii))**.** *Every morphism g in* Gal *is of the form $g = g_1.\Delta^{-n} = \Delta^{-n}.g_2$ where g_1 and g_2 are morphisms in* Gal_+*.*

Proof. Any $g \in \mathrm{Hom}_{\mathrm{Gal}}(C_0, C_k)$ can be decomposed in the following form

$$g = g_1 h_1^{-1} g_2 h_2^{-1} \cdots g_k h_k^{-1}$$

where g_i and h_i are morphisms in Gal_+. Write $h_i = u(B_0, B_1) \cdots u(B_{n_i-1}, B_{n_i})$. Observe that

$$(B_j).\Delta = u(B_j, -B_j) = u(B_j, B_{j+1})u(B_{j+1}, -B_j) \implies u(B_j, B_{j+1})^{-1} = u(B_{j+1}, -B_j).\Delta^{-1}$$

for all $j \in \{0, \dots, n_i - 1\}$. Hence, by Lemma 61

$$\begin{aligned} h_i^{-1} &= u(B_{n_i-1}, B_{n_i})^{-1} \cdots (B_1, B_2)^{-1}(B_0, B_1)^{-1} \\ &= (u(B_{n_i}, -B_{n_i-1}).\Delta^{-1}) \cdots (u(B_2, -B_1).\Delta^{-1})(u(B_1, -B_0).\Delta^{-1}) \\ &= u(B_{n_i}, -B_{n_i-1})u(-B_{n_i-1}, B_{n_i-2}) \cdots (-1)^{n_i-2}u(B_2, -B_1)(-1)^{n_i-1}u(B_1, -B_0).\Delta^{-n_i}. \end{aligned}$$

That is, we can write $h_i^{-1} = g_i'.\Delta^{-n_i}$ for some morphism g_i' in Gal_+. Thus, letting $n = \sum_i n_i$

$$g = g_1 h_1^{-1} g_2 h_2^{-1} \cdots g_n h_n^{-1} = g_1(g_1'.\Delta^{-n_1}) \cdots g_k(g_k'.\Delta^{-n_k}) = \widetilde{g}.\Delta^{-n}.$$

$\square$

6. An outline of Deligne's theorem

In this chapter we outline Deligne's proof of the following theorem.

Theorem 63 (Deligne [10], Main Theorem)**.** *Let* V *be a finite dimensional real vector space,* $\mathcal{H}$ *a finite set of hyperplanes of* V*, and* $Y := V^{\mathbb{C}} - \bigcup_{H\in\mathcal{H}} H^{\mathbb{C}}$*. Suppose that the connected components of* $V - \bigcup_{H\in\mathcal{H}} H$ *are open simplicial cones. Then* Y *is* $K(\pi, 1)$*.*

Since Y is path-connected, locally path-connected, and semilocally simply connected, there is a universal covering space $\widetilde{Y}$ whose points may be realised as homotopy classes of paths in Y starting at a basepoint. The goal is to show that $\widetilde{Y}$ is contractible, and so by the isomorphism $\pi_i(Y) \simeq \pi_i(\widetilde{Y}) \simeq 0$ for all $i \geq 2$, we prove the second component of the $K(\pi, 1)$ conjecture for Artin braid groups of finite type. The proof proceeds through the following steps:

1. By the path-gallery analogy discussed in Chapter 5, we construct a combinatorial model, called the **Deligne complex** I, of the universal covering space $\widetilde{Y}$, from galleries out of a fundamental chamber.
2. We show that the Deligne complex I is a skeleton of a contractible CW complex $\hat{I}$.
3. We construct the universal covering space $\widetilde{Y}$ from I.
4. By a clever choice of an open covering U of $\widetilde{Y}$, we relate $\hat{I}$ and $\widetilde{Y}$ via the nerve complex $N(U)$.
5. Contractibility of $\widetilde{Y}$ follows from contractibility of $\hat{I}$.

In section §6.2 we illustrate a method for showing that a simplicial complex deformation retracts onto a subcomplex by a process called **simplicial collapse**. In section §6.5 we provide an outline of Leray's theorem which tells us when a **nerve complex** of a cover is simplicially isomorphic to the simplicial complex it covers.

For a more detailed discussion of the combinatorial aspects of Deligne's argument see [11].

6.1. The Deligne complex I.

We want a model of the universal covering space $\widetilde{Y}$ built up from contractible building blocks. The hope is that such a model can then be shown to be contractible. Given a simplicial arrangement $(V, \mathcal{H})$ and $n := \dim V$, the hyperplanes cut out a cellular decomposition of $\mathcal{S} := S^{n-1}$ (see Figure 41). We call the induced arrangement on $\mathcal{S}$ the **spherical arrangement** $(\mathcal{S}, \mathcal{H})$.

We may carry over the terminology developed for $(V, \mathcal{H})$ to the spherical arrangement $(\mathcal{S}, \mathcal{H})$ where the facets of the latter are determined by the inclusion $\mathcal{S} \hookrightarrow V$. Galleries, adjacency, chambers, and other terminology are similarly induced by this inclusion. In particular, since every positive-dimensional facet of $(V, \mathcal{H})$ meets exactly one facet of $(\mathcal{S}, \mathcal{H})$, we have the correspondence

$$\left\{ \begin{array}{c} \text{Positive dimensional facets of} \\ (V, \mathcal{H}) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Facets of} \\ (\mathcal{S}, \mathcal{H}) \end{array} \right\}.$$

The order complex of the face poset of $(\mathcal{S}, \mathcal{H})$ is an abstract simplicial complex. The facets are thus identified with simplexes in $(\mathcal{S}, \mathcal{H})$. These cells will form the building blocks of our model.

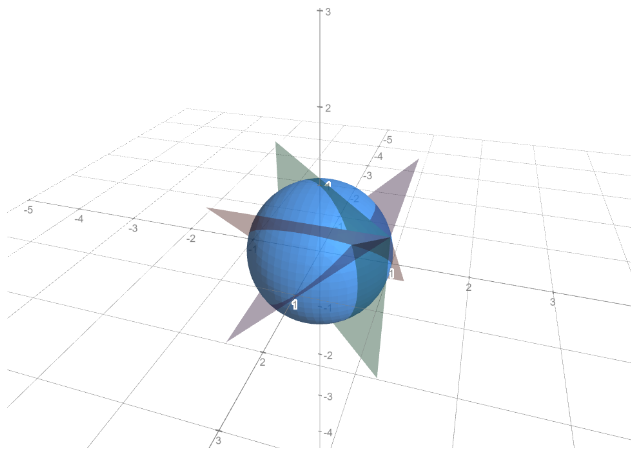


FIGURE 41. Cellular decomposition of S^2 by simplicial arrangement in $\mathbb{R}^3$.

To construct a model of $\widetilde{Y}$ we first need some way to represent the homotopy classes of paths in Y starting at some basepoint. In the path-gallery analogy discussed in Chapter 5, choosing a basepoint amounts to choosing a fundamental chamber A_0, and paths out of this basepoint can be regarded as galleries starting at A_0 in Gal. Let us consider a motivating example of constructing a space built from galleries out of the spherical arrangement in Figure 42.

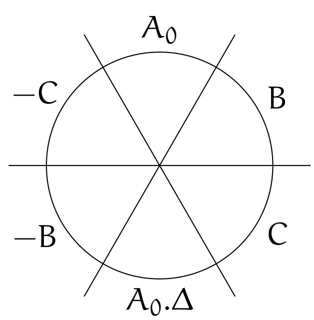


FIGURE 42. Spherical arrangement for $\dim V = 2$.

Consider the following galleries starting at A_0:

$$g_1 := u(A_0, C), \quad g_2 := u(A_0, -B), \quad g_3 := \Delta, \quad g_4 := (A_0, B, A_0).$$

We take the closure of the chambers in $(\mathcal{S}, \mathcal{H})$ as our building blocks and for each gallery we construct a simplicial complex via a sequence of gluings indexed by the galleries starting from A_0. Figure 43 depicts such simplicial complexes associated to the above galleries.

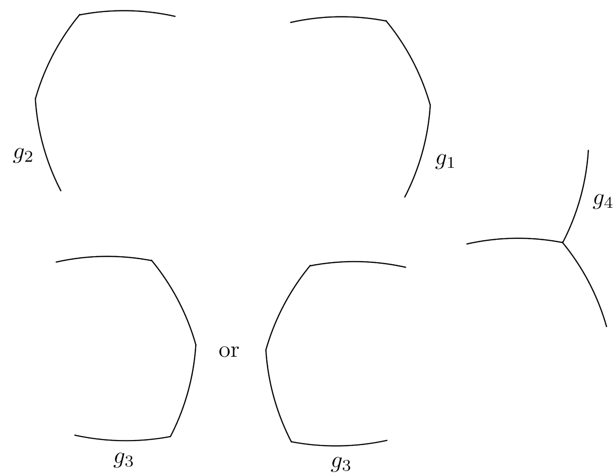


FIGURE 43. Construction of simplicial complexes associated to g_1, g_2, g_3 and g_4.

We combine these complexes into a single complex whose chambers encode the galleries g_1, g_2, g_3 and g_4 as well as their positive subgalleries (Figure 44).

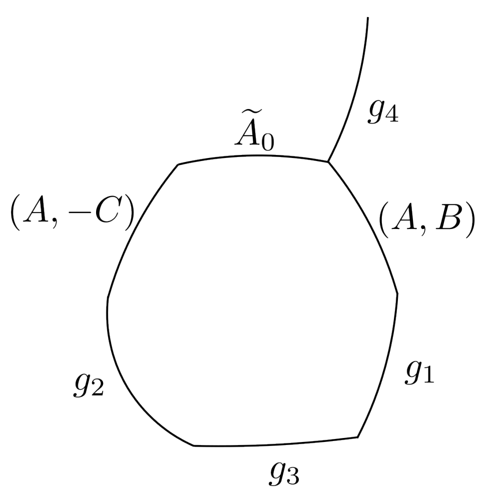


FIGURE 44. A simplicial complex whose cells are indexed by positive galleries.

We now formalise this exercise by constructing the **Deligne complex** I of $(V, \mathcal{H})$ centred at the fundamental chamber A_0. Our complex will be indexed by the galleries out of A_0

$$\mathcal{Z} := \coprod_{\substack{B \text{ chamber} \\ \text{of } (V,\mathcal{H})}} \mathrm{Hom}_{\mathrm{Gal}}(A_0, B).$$

For each $g \in \mathcal{Z}$, we define $\mathrm{tar}(g)$ to be the closure of the chamber in $(\mathcal{S}, \mathcal{H})$ corresponding to the target chamber of g. We now consider the disjoint union of these complexes

$$Z := \bigsqcup_{g \in \mathcal{Z}} \mathrm{tar}(g).$$

There is a canonical projection map $q' : Z \to \mathcal{S}$. We thus topologise Z with the initial topology so that q' is continuous. We can then construct I in the following way: if $g \in \mathcal{Z}$ and C is a chamber in $\mathcal{S}$ adjacent to $q'(\mathrm{tar}(g))$, we glue $\mathrm{tar}(g)$ with $\mathrm{tar}(gu(q'(\mathrm{tar}(g)), C))$ by identifying the points which agree in the image of q'. Since q' is continuous and invariant with respect to the identified points of the gluing, we obtain a continuous map $q : I \to \mathcal{S}$ such that the following diagram commutes

$$\begin{tikzcd} Z \arrow[r, "q'"] \arrow[d, two heads] & \mathcal{S} \\ I \arrow[ur, "q"'] & \end{tikzcd} .$$

We note that I inherits a natural simplical complex structure from its components in $\mathcal{S}$ and so again there is a natural way to carry over the terminology of facets, chambers, et cetera, on the Deligne complex. We let $\widetilde{A}_0$ denote the chamber in I indexed by the gallery (A_0). Given a gallery g with source A_0 and target A, we index the chamber in I corresponding to g as $\widetilde{A} = \widetilde{A}_0.g$. Then if h is a gallery with source A, we can write $\widetilde{A}.h = \widetilde{A}_0.gh$ which can be thought of as a change of fundamental chamber.

The Deligne complex I is built up by infinitely many cells. It is convenient to consider the chain of subcomplexes defined by

$$(I)_n := \bigcup_{\ell(g) \leq n} \widetilde{A}_0.g.$$

We thus have a direct system of CW-complexes under cellular inclusions

$$(I)_0 \hookrightarrow (I)_1 \hookrightarrow (I)_2 \hookrightarrow \cdots$$

with direct limit

$$I = \varinjlim (I)_n.$$

Example 64. Consider the following spherical arrangement (Figure 45).

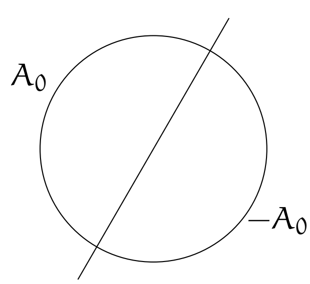


FIGURE 45. Spherical arrangement cut out by one hyperplane.

Then the Deligne complex I of this arrangement centred at A_0 is built up of the subcomplexes $(I)_n$ where $n \in \mathbb{N}$ (Figure 46).

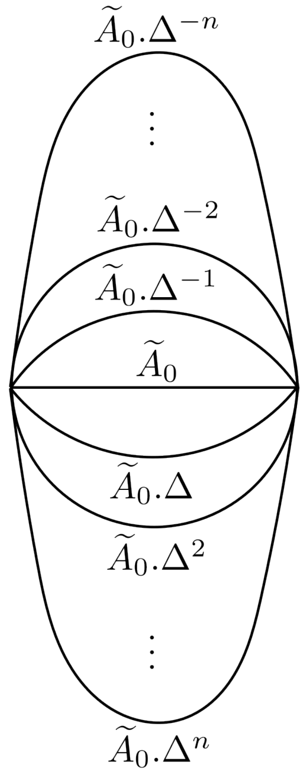


FIGURE 46. $(I)_n$ of spherical arrangement in Figure 45.

By contracting $(I)_n$ along the line segment labelled by $\widetilde{A}_0$, we see that

$$(I)_n \simeq \bigvee_{i=-n}^{n} \mathcal{S}.$$

That is, $(I)_n$ is a bouquet of spheres.

Let A be a chamber of I. Then there exists a gallery with source A_0 such that $A = \widetilde{A}_0.g$. We define the sphere at A by

$$S(A) := \bigcup_{\substack{B \text{ chamber} \\ \text{of } (V,\mathcal{H})}} \operatorname{tar}(gu(qA, B)).$$

One can verify that $q(S(A))$ is simplicially isomorphic to $\mathcal{S}$. This makes intuitive sense: each chamber in I can be thought of as the fundamental chamber of a copy of the spherical arrangement $(\mathcal{S}, \mathcal{H})$. As such, for each chamber A, there is a corresponding sphere in I denoted by $S(A)$, which is the boundary of some ball, we denote by $b(A)$. Thus, we can attach the balls $\{b(A) : A \text{ is a chamber of } I\}$ to the Deligne complex in the natural way, and obtain the simplicial complex $\hat{I}$. We define the natural projection $\hat{q} : \hat{I} \to B^n$, where B^n is the n-ball, homeomorphic to $b(A)$.

Example 65. Let $(I)_n$ be the Deligne complex from Example 64, which is a bouquet of $2n$ spheres. By attaching 2-cells along the boundary of these spheres, we obtain the complex $(\hat{I})_n$ which is the wedge sum of $2n$ disks. Each of these disks are contractible, and so $(I)_n$ is contractible.

6.2. **Simplicial collapse.** One of the main tasks in Deligne's argument is to show that the simplicial complex $\hat{I}$ is contractible. His argument is essentially an argument about the collapsibility of $\hat{I}$. In this section we define what it means for a simplicial complex to be collapsible, and show that it is a sufficient condition for contractibility.

Definition 66. Let K be an abstract simplicial complex with simplex τ. We say that τ is a **free face** of K if there exists a simplex σ in K such that

(i) $\tau \subsetneq \sigma$; and
(ii) σ is a maximal face of K and no other maximal face of K contains τ.

Definition 67. A **simplicial collapse** of K is the removal of all simplices γ such that

$$\tau \subseteq \gamma \subseteq \sigma$$

where τ is a free face. If a simplicial complex K_1 collapses to a simplicial complex K_2, we denote the collapse by $K_1 \searrow K_2$. If additionally we have $\dim \tau = \dim \sigma - 1$, then the collapse is called an **elementary collapse**.

Example 68. Consider the following simplicial collapse of a 1-cell (Figure 47).

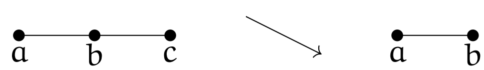


FIGURE 47. $(\sigma, \tau) = (ab, c)$ elementary collapse of two 1-simplices joined together

Example 69. Consider a simplicial collapse of a filled-in box (Figure 48).

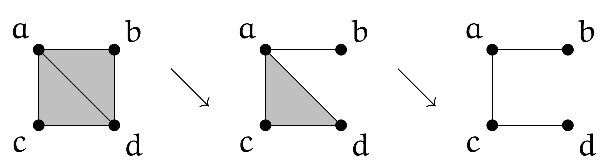


FIGURE 48. $(\sigma_1, \tau_1) = (abd, bd)$ and $(\sigma_2, \tau_2) = (acd, ad)$ collapse of box.

The main tool we will use to explain Deligne's argument, is the following.

Theorem 70. *If two simplicial complexes* $K_2 \subseteq K_1$ *differ by a sequence of elementary collapses, then* $|K_2|$ *is a deformation retract of* $|K_1|$.

Proof. See Proposition 6.14 in 'Combinatorial Algebraic Topology' by Dmitry Kozlov [20]. □

Definition 71. If K collapses to a single point, then we say K is **collapsible** and so $|K|$ is contractible.

Example 72. Since a square (not filled in) does not deformation retract to a point, the square is an example of a simplicial complex which is not collapsible. Examples 68 & 69 are both collapsible.

6.3. **Contractibility of $\hat{I}$.** In this section, we outline the proof by Deligne that the complex $\hat{I}$ is contractible. To do this, we first introduce the **positive Deligne complex** I_+ which is constructed in an identical way to I except over the category of positive galleries Gal_+. On this simplicial complex, we have the following technical lemma.

Lemma 73 (Deligne [10], Lemma 2.9.2)**.** *Let* $A := \widetilde{A}_0.g$ *be in* $(I_+)_{n+1}$ *but not* $(I_+)_n$*. Suppose there exists a chamber* $C := \widetilde{A}_0.h$ *in* $(I_+)_{n+1}$ *distinct from* A *sharing a facet* F*. Then there exists another chamber* $B = \widetilde{A}_0.h'$ *in* $(I_+)_n$ *and a wall* M *of* qB*, containing* qF*, such that* $A = B.\Delta(M)$*.*

Proof. See Deligne's paper. Note, the proof of this lemma does not depend on Deligne's lemma 2.9.1, despite preceding its statement in his paper. ☐

Proposition 74 (Deligne [10], Lemma 2.9.1)**.** $(\hat{I}_+)_n$ *is a deformation retract of* $(\hat{I}_+)_{n+1}$*, for* $n \geq 0$*.*

Proof sketch. We need to construct a continuous family of continuous maps $\{\varphi_t\}_{0\leq t\leq 1}$ such that $\varphi_t : (\hat{I}_+)_{n+1} \to (\hat{I}_+)_{n+1}$ where $\varphi_0 = \mathrm{id}$, $\varphi_1((\hat{I}_+)_{n+1}) = (\hat{I}_+)_n$ and $\varphi_t|_{(\hat{I}_+)_n} = \mathrm{id}$. By the pasting lemma in topology, it is sufficient to specify φ_t by where it sends the closed chambers and balls of $(\hat{I}_+)_{n+1}$, so long as the map agrees on their intersections. Let $A := \widetilde{A}_0.g$ be a chamber in $(\hat{I}_+)_{n+1}$ but not in $(\hat{I}_+)_n$. There are two cases.

(Case 1): $(\overline{A} \cap (\hat{I}_+)_n) \neq \partial\overline{A}$. Let M and N be walls of A (not necessarily distinct) such that M and N are in $(\hat{I}_+)_n$. Since qA is a chamber of a simplicial arrangement, qM and qN share a facet (Proposition 56). Hence, the subset of $\partial\overline{A}$ contained in $(\hat{I}_+)_n$ is a connected subcomplex. Since $\overline{A}$ is realised by the order complex of the poset of faces of $q\overline{A}$, we can construct a sequence of elementary collapses, collapsing $\overline{A}$ onto the connected subcomplex of $\partial\overline{A}$ in $(\hat{I}_+)_n$. We give an illustrative example of such a sequence for a chamber of a simplicial arrangement of the 2-sphere (Figures 49). The walls of A in $(\hat{I}_+)_n$ are marked in red.

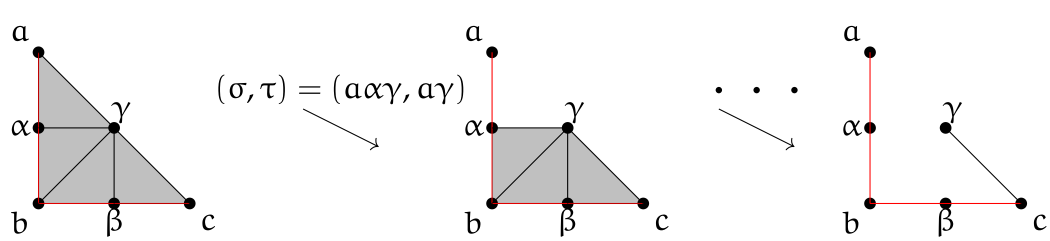


FIGURE 49. Elementary collapses of the maximal faces in A.

Then $\overline{A}$ collapses onto $(\hat{I}_+)_n$ by a final elementary collapse of the form $(\sigma, \tau) = (\gamma c, \gamma)$. By Theorem 70, $\overline{A}$ deformation retracts onto $\overline{A} \cap (\hat{I}_+)_n$, and so we denote the corresponding continuous family of maps by $\{(\varphi|_{\overline{A}})_t\}_{0 \le t \le 1}$.

(Case 2): $(\overline{A} \cap (\hat{I}_+)_n) = \partial\overline{A}$. For any wall M of A, there is a distinct chamber C in $(I_+)_{n+1}$ sharing M. By Lemma 73, there exists another chamber $B_M = \widetilde{A}_0.h_M$ in $(I_+)_n$ such that B_M contains the wall M. Hence, g^* begins with $(A, A.\Delta(M))$ for any wall M of A. By Proposition 59, there exists a unique chamber Δ in $\mathcal{S}$ such that g ends with $u(\Delta, qA)$ and $\mathcal{H}(\Delta, qA) = \mathcal{H}(\Delta, B_M) \sqcup \mathcal{H}(B_M, qA)$. That is, any gallery from $\widetilde{A}_0$ to A must "bottleneck" through Δ. Letting $B := \widetilde{A}_0.u(A_0, \Delta)$, it follows that

$$\overline{A} \subseteq S(B) = \bigcup_{C \text{ chamber in } \mathcal{S}} \overline{Bu(\Delta, C)} \simeq \mathcal{S}.$$

Thus $S(B)$ is the boundary of an open ball $b(B)$ in $(\hat{I}_+)_{n+1}$. It thus remains to show that $S(B) - A$ is a deformation retract of $\overline{b(B)}$. This is an example of cellular collapse (see [20]), we provide an illustrative example of such a collapse (Figure 50).

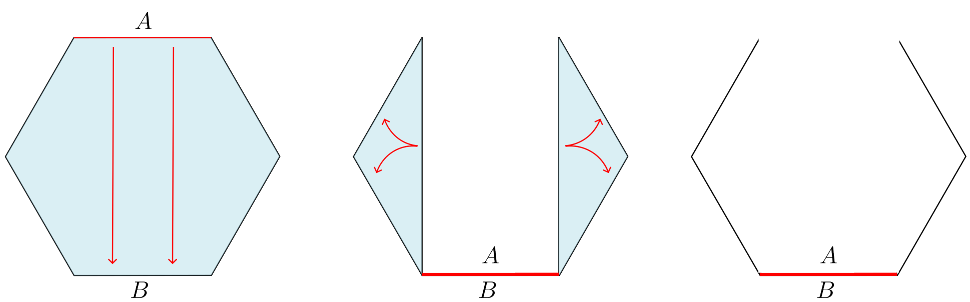


FIGURE 50. Example of $\overline{b(B)} \searrow S(A) - A$.

We denote the corresponding continuous family of maps by $\left\{\left(\varphi|_{\overline{b(B)}}\right)_t\right\}_{0 \le t \le 1}$.

Since these deformation retracts of closed subsets of $(\hat{I}_+)_{n+1}$ are fixed on connected segment of their boundaries in $(\hat{I}_+)_n$, it follows by the pasting lemma that there is a deformation retract of $(\hat{I}_+)_{n+1}$ onto $(\hat{I}_+)_n$. ☐

We may now apply this proposition inductively to obtain the following theorem.

Theorem 75 (Delgine [10], Proposition 2.9)**.** *The simplicial complex $\hat{I}_+$ is contractible.*

Proof. Since $(\hat{I}_+)_0$ is simplicially isomorphic to a chamber in $\mathcal{S}$, $(\hat{I}_+)_0$ is contractible. By applying Proposition 74 inductively, we see that $(\hat{I}_+)_n$ is contractible for all $n \in \mathbb{N}$. Given any continuous map $f : S^k \to \hat{I}_+$, $f(S^k)$ is a compact subspace of $\hat{I}_+$. Since $\hat{I}_+$ is a CW complex, $f(S^k)$ is contained in a finite subcomplex of $\hat{I}_+$ (see Hatcher Proposition A.1 [15]). This implies that there exists $n \in \mathbb{N}$ such that $f(S^k) \subseteq (\hat{I}_+)_n$. By contractibility of $(\hat{I}_+)_n$, it follows that $\hat{I}_+$ is weakly contractible. By Whitehead's theorem in algebraic topology, this implies $\hat{I}_+$ is homotopy equivalent to a point, and thus contractible. ☐

This result can be extended to the $\hat{I}$.

Corollary 76 (Deligne [10], Proposition 2.14 & Theorem 2.15)**.** *The complex $\hat{I}$ is contractible.*

6.4. **Construction of a covering space $\widetilde{Y} \to Y$.** The complex $\hat{I}$ provides a combinatorial model of galleries in Y. We need to relate this model to the paths of Y. In other words, we want to construct the universal covering space of $\widetilde{Y}$ of Y using the machinery already constructed for $\hat{I}$.

A "walk" in the Deligne complex I is a gallery of the form $(\Delta(M_1)^{\varepsilon_1}, \Delta(M_2)^{\varepsilon_2}, \dots, \Delta(M_n)^{\varepsilon_n})$ where $\varepsilon_i \in \{\pm 1\}$. We can associate this gallery to a path in Y, whose image under $\mathfrak{R}$ passes from chamber to chamber, crossing the successive walls $M_1, M_2, \dots, M_n$ at points $m_1, m_2, \dots, m_n$. For each $1 \leq i \leq n$, with $\varepsilon_i = 1$ (resp $\varepsilon_i = -1$) the path passes through the component R of $\mathfrak{R}^{-1}(m_i) \cap Y$ (which is in bijection with $V - M_i$) such that $\mathfrak{I}R$ contains (resp does not contain) the previous chamber (Figure 51).

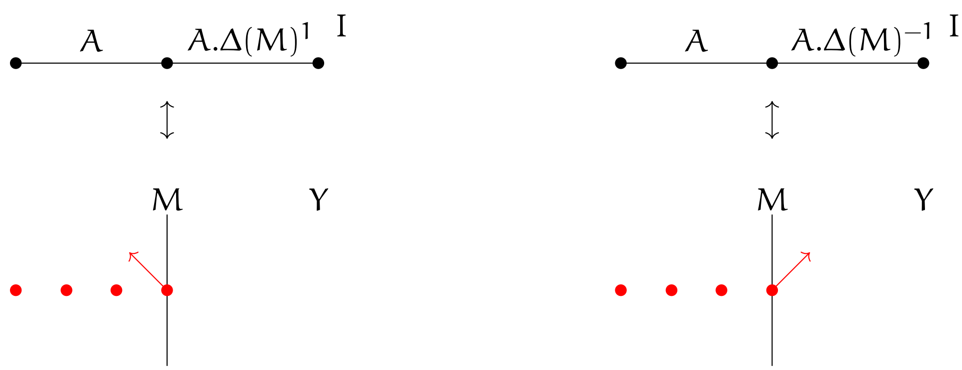


FIGURE 51. Galleries in I correspond to paths in Y_M.

To construct the universal covering space we want to pick out overlapping open sets of Y which form the sheets of the covering space. To do this, we must develop some more notation.

Lemma 77. *Let A, B and C be three chambers of I with $C \subset S(A) \cap S(B)$. Let M be a hyperplane separating qA and qB, and let $D'_M(qC)$ be the closed half space of V with respect to M containing qC. Then q induces a simplicial isomorphism*

$$S(A) \cap S(B) \simeq \bigcap_{M \in \mathcal{H}(qA,qB)} \mathcal{S} \cap D'_M(qC).$$

Example 78. Recall that the Deligne complex I centred at A_0 of the simplicial arrangement in Figure 52 (left) has as a subcomplex, the construction in Figure 52 (right).

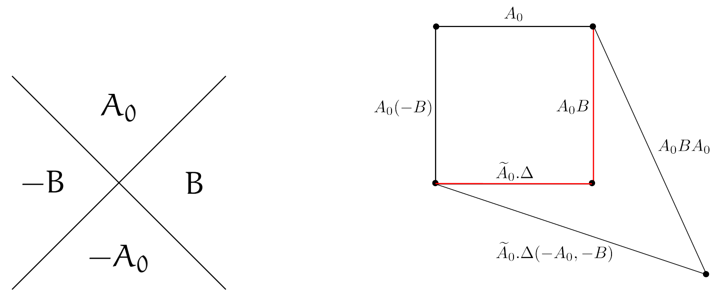


FIGURE 52. The subcomplex $S(A) \cap S(B)$ of I (shaded in red).

Let C be a chamber of $(V, \mathcal{H})$. We define the open set

$$Y(C) := \{x \in V^{\mathbb{C}} : \mathrm{pr}_F \mathfrak{I}(x) \in \pi_F'(C), \ F \text{ the unique facet s.t } \mathfrak{R}(x) \in F\}.$$

Figure 54 depicts some of the elements in $Y_{I_2(3)}(C)$.

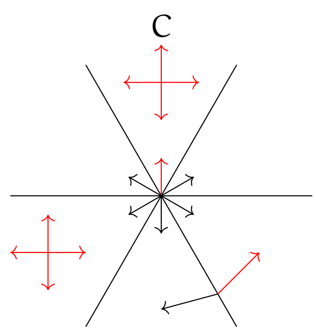


FIGURE 53. Elements of $Y(C)$ in red, elements not in $Y(C)$ in black.

Lemma 79. *For any chambers A and B in I. Then if $C, C' \in S(A) \cap S(B)$ we have*

$$\bigcap_{M \in \mathcal{H}(qA,qB)} D_M'(qC) = \bigcap_{M \in \mathcal{H}(qA,qB)} D_M'(qC').$$

Proof. By assumption, there exists $C \in S(A) \cap S(B)$ and so there exists minimal galleries g_1 and g_2 such that

$$C = A.g_1 = B.g_2 \implies B = A.(g_1 g_2^{-1}).$$

Let $\widetilde{g} := g_1 g_2^{-1}$. Then

$$S(A) \cap S(B) = \bigcup_{\substack{h \text{ a minimal gallery s.t.} \\ h \text{ begins with } \widetilde{g}}} \overline{A.h}.$$

Thus given any two $C, C' \in S(A) \cap S(B)$, it follows by minimality that

$$\mathcal{H}(qA, qB) \cap \mathcal{H}(qC, qC') = \varnothing.$$

So for any $M \in \mathcal{H}(qA, qB)$, C and C' are on the same side of M. $\square$

Let A and B be chambers in I. If $S(A) \cap S(B)$ contains a chamber C, we define

$$Y'(A,B) := \bigcap_{M \in \mathcal{H}(qA,qB)} = D'_M(qC)$$

which is well-defined by Lemma 79. We then define

$$Y(A,B) := Y(qA) \cap Y(qB) \cap \mathfrak{R}^{-1} Y'(A,B).$$

Example 80. Note that $Y(A,A) = Y(qA) \cap \mathfrak{R}^{-1} V = Y(qA)$ (see Example 78). Consider the open set $Y_{I_2(3)}(A,B)$ for the following choices of B (see Figure).

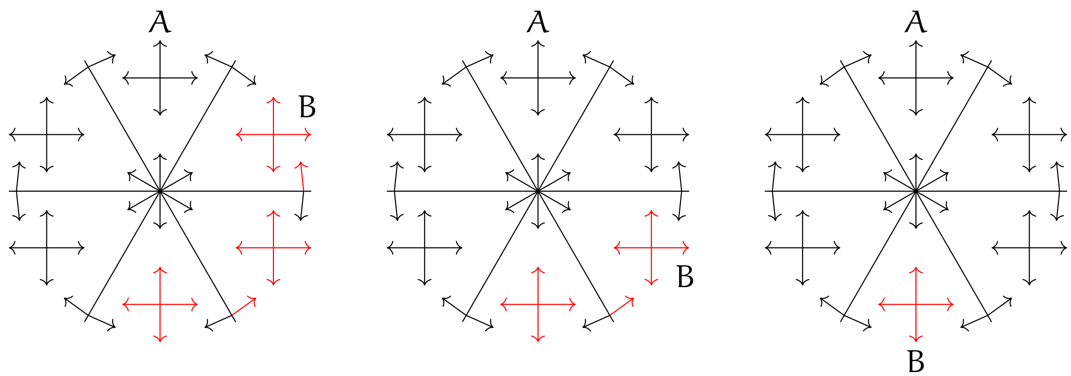


FIGURE 54. Elements of $Y_{I_2(3)}(A,B)$.

This allows us to construct the universal covering space $\widetilde{Y}$ in the following way: we construct our building blocks

$$Y' := \bigsqcup_{b(A) \text{ balls of } \hat{I}} Y(qA)$$

where we denote the component corresponding to $b(A)$ by $Y'(b(A))$. There is a natural projection $p' : Y' \to Y$. We then construct the universal covering space

$$\widetilde{Y} := Y' /_{\sim}$$

where $x \sim y$ if and only if $p'(x) = p'(y)$ and

$$p'(x), p'(y) \in Y(A,B) \quad \text{where } x \in Y'(b(A)) \text{ and } y \in Y'(b(B)).$$

We thus have the natural projections $p : \widetilde{Y} \to Y$ and $\pi : Y' \to \widetilde{Y}$. For each ball $b(A)$, we define

$$\widetilde{Y}(b(A)) := \pi(Y'(b(A)).$$

Theorem 81 (Deligne [10], Theorem 3.7.A)**.** *The covering space $p : \widetilde{Y} \to Y$ is a universal covering space.*

Proof. See Deligne's paper. □

6.5. **Leray's nerve theorem.** In this section, we introduce Leray's theorem which is the machinery that can used to bridge the connection between the combinatorial model $\hat{I}$ and the universal covering space $\widetilde{Y}$.

Definition 82. Let X be a topological space, with open cover $\mathcal{U} := \{U_s\}_{s \in S}$. The **nerve complex** of $\mathcal{U}$ is the abstract simplicial complex

$$N(\mathcal{U}) := \left\{ J \subseteq S : \bigcap_{j \in J} U_j \neq \emptyset \text{ and } |J| < \infty \right\}.$$

Example 83. Consider the open covering $\mathcal{U}$ of S^1 by an upper and lower semicircle (Figure 55).

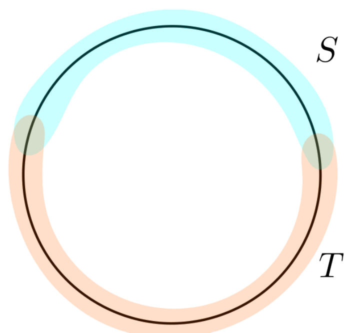


FIGURE 55. Open covering of S^1 by semicircles.

The nerve complex of $\mathcal{U}$ is

$$N(\mathcal{U}) = \{\{S\}, \{T\}, \{S, T\}\}$$

which has a geometric realisation homotopy equivalent to a 1-simplex (Figure 56).

FIGURE 56. Geometric realisation of $N(\{S, T\})$.

Example 84. Consider the following covering $\mathcal{U}$ of S^1, by three open sets that each intersect (Figure 57).

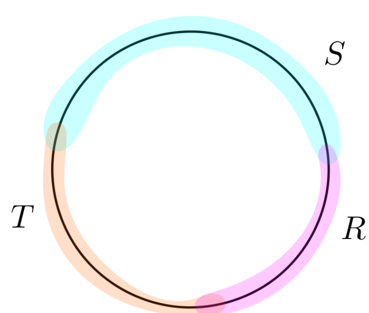


FIGURE 57. Open covering of S^1 by three open sets.

Then, we have the nerve complex

$$N(U) = \{\{S\},\{T\},\{R\},\{S,T\},\{S,R\},\{T,R\}\}$$

and thus has a geometric realisation homotopy equivalent to S^1 (Figure 58).

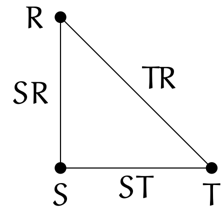


FIGURE 58. Geometric realisation of $N(\{S,T,R\})$.

Examples 83 & 84 illustrate that $|N(U)|$ may or may not be homotopy equivalent to X. A natural question to ask is when a nerve complex $N(U)$ can serve as a simplicial model of X, in the sense that $|N(U)| \simeq X$. Leray's theorem gives a sufficient condition.

Theorem 85 (Leray)**.** *Let* $U := \{U_s\}_{s\in S}$ *be an open covering of* X*. If* $\bigcap_{j\in J} U_j$ *is contractible, for all* $J \in N(U)$*, then* $|N(U)|$ *is homotopy equivalent to* X*.*

Remark 86. In Example 83, $|N(U)|$ has the homotopy type of a point, and so by Leray's theorem we should expect that $S \cap T$ is not contractible, which we saw is the case. On the other hand, in Example 84 each non-empty intersection of open sets in U is contractible, and hence by Leray's theorem we should expect $|N(U)|$ to be homotopy equivalent to S^1, which again, we observed was the case.

6.6. **Contractibility of $\widetilde{Y}$.** In this final section, we relate $\widetilde{Y}$ to $\hat{I}$ using Leray's theorem, thus proving that $\widetilde{Y}$ is contractible

$$\hat{I} \xleftarrow[\simeq]{\text{(Prop 87)}} |N(U)| \xrightarrow[\simeq]{\text{(Thm 85)}} \widetilde{Y}\ .$$

We construct an open cover U of $\widetilde{Y}$ satisfying Leray's theorem such that $\hat{I}$ is the nerve complex of U. It follows that $\widetilde{Y} \simeq \hat{I}$ and so $\widetilde{Y}$ is contractible. Let F be a facet of the simplicial complex $\hat{I}$, then we define the set

$$\widetilde{Y}(F) := \bigcup_{\substack{F\subset S(C) \\ C \text{ is a chamber of } I}} \widetilde{Y}(b(C)) \cap p^{-1}\mathfrak{R}^{-1}\mathrm{Et}(qF)$$

where $\mathrm{Et}(qF)$ is the **star** (or **étoile** in French) of the facet qF consisting of the union of facets E in $(V,\mathcal{H})$ such that $qF \subset \overline{E}$. We obtain the following important results.

Proposition 87 (Deligne [10], Lemma 3.7.4.(i))**.** *The collection of sets*

$$U = \{\widetilde{Y}(s) : s \textit{ is a vertex of the simplical complex } \hat{I}\}$$

forms an open covering of $\widetilde{Y}$*, whose nerve complex* $N(U)$ *is simplicially isomorphic to* $\hat{I}$

Proof. See Deligne's paper. $\square$

Lemma 88 (Deligne [10], Lemma 3.7.4.(ii))**.** *Let* $s_0, \dots, s_p$ *be the vertices of the facet* F *of* I*, then*

$$\widetilde{Y}(F) = \widetilde{Y}(s_0) \cap \cdots \cap \widetilde{Y}(s_p).$$

Proof. See Deligne's paper. $\square$

We now let Y_F be the complexified hyperplane complement of $(V/P, \mathcal{H}_P)$ where $P = \operatorname{supp} F$. We can relate via homotopy equivalence the universal cover $\widetilde{Y}_F$ of Y_F, constructed as in §6.4, with an intersection of open sets in U.

Lemma 89 (Deligne [10], Lemma 3.8)**.** *For any facet* F *of* I*,*

$$\widetilde{Y}_F \simeq \widetilde{Y}(F).$$

Proof. See Deligne's paper. $\square$

Theorem 90. *The universal covering space* $\widetilde{Y}$ *is contractible.*

Proof sketch. We proceed inductively on the the dimension of V. If $\dim V = 1$, then the simplicial arrangement is of the form 59.

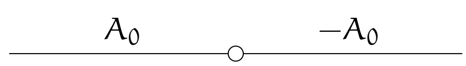


FIGURE 59. 1-dimensional simplicial arrangement.

Then $Y \simeq \mathbb{C} - \{0\}$ whose universal covering space is $\widetilde{Y} \simeq \mathbb{C}$ which is contractible. Assume that for all simplicial arrangements satisfying $\dim V \leq r$ where $r > 1$, the space $\widetilde{Y}$ is contractible. Now consider any simplicial arrangement where $\dim V = r + 1$. For any facet F of $\widehat{I}$ (which will never correspond to a zero-dimensional facet in $(V, \mathcal{H})$), it follows by the inductive hypothesis that

$$\widetilde{Y}_F \overset{\text{(Lem 89)}}{\simeq} \widetilde{Y}(F)$$

is contractible. But by Lemma 88, it follows that U satisfies the criterion for Leray's theorem. Therefore

$$\widetilde{Y} \overset{\text{(Thm 85)}}{\simeq} |N(U)| \overset{\text{(Prop 87)}}{\simeq} \widehat{I}$$

and so $\widetilde{Y}$ is contractible by Theorem 75. $\square$

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